CS40 Winter 2021 Exercises

March 6, 2021

1. Consider two functions $f : A \to A$ and $g : A \to A$. Assume now that f is injective and that g is surjective. What properties does the function $g \circ f$ have if
   (a) A is a finite set?
   (b) A is an infinite set?
2. A monkey has three mangos, two papayas, and two kiwi fruits. If the monkey eats one piece of fruit each day, and only the type of the fruit matters,
   (a) in how many different ways can these fruits be consumed?
   (b) in how many different ways can these fruits be consumed if he never eats papayas in consecutive days?
3. Consider the identity $\binom{2n}{n} \leq 2^{2n}$, for an integer $n \geq 0$.
   (a) Prove by induction.
   (b) Prove by giving a combinatorial argument. Thoroughly explain your argument.
4. Prove
$$\frac{(2n)!}{n!n!} \geq \frac{4^n}{2\sqrt{n}}$$
   for $n \geq 1$.
5. Let L be the set of all the straight lines in the plane $\mathbb{R}^2$.
   Let $\|$ and $\perp$ be the following relations over L:
$$\| = \{(\mathscr{L}_1, \mathscr{L}_2) : \mathscr{L}_1 \text{ is parallel to } \mathscr{L}_2\}$$
$$\perp = \{(\mathscr{L}_1, \mathscr{L}_2) : \mathscr{L}_1 \text{ is perpendicular to } \mathscr{L}_2\}$$
   (a) Prove $\|$ is an equivalence relation.
   (b) Prove $\perp$ is *not* an equivalence relation.
   (c) Prove that $H = (\perp) \cup (\|)$ is an equivalence relation and explain its equivalence classes.

6. At Very Secure Corporation Inc., computer passwords are required to have four numeric digits (out of 10 possible), six letters (of 26 possible) and two special characters (of 20 possible). Letters and special characters may repeat, however the digits are required to be distinct. How many different possible passwords are there?

7. A, B, C are some sets. Prove or disprove:

   (a) $(A \cap B^c) \subset A$

   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   (c) $A = B$ follows from $A \cup B = A$ and $A \cap B = A$

   (d) $\left|2^A\right| > 1$

8. Let $A = \{-4, -2, -1, 0, 1, 2, 4\}$, what is the cardinality of the set $\left(A^2 - \mathbb{N}^2\right)$? (remember $A^2 = A \times A$)

9. Let $\triangle$ denote the symmetric set difference, that is $A \triangle B = (A \backslash B) \cup (B \backslash A)$.

   Prove: If $\mathcal{P}(A \triangle B) \subseteq \mathcal{P}(A) \triangle \mathcal{P}(B)$, then $A \subseteq B$ or $B \subseteq A$.

10. Let $n > 1$ integer. Prove that $n^4 + 4^n$ is not prime.

    [Hint: $a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$, which known as "Sophie Germain's identity".]

11. Let $A_n = 1^2 2^1 + 2^2 2^2 + 3^2 2^3 + \ldots + n^2 2^n = \sum_{k=1}^{n} k^2 2^k$, defined for $n \in \mathbb{Z}^+$. Prove using induction

$$A_n = \left(2n^2 - 4n + 6\right) 2^n - 6$$

12. Prove that $n^3 + 2n$ and $n^4 + 3n^2 + 1$ are relatively prime.

13. Show that there do not exist four consecutive (non-zero) binomial coefficients $\binom{n}{k}, \binom{n}{k+1}, \binom{n}{k+2}, \binom{n}{k+3}$ in arithmetic progression.

14. We define an *ideal* $\mathfrak{I}$ of $\mathbb{Z}$ as a non-empty subset $\mathfrak{I} \subseteq \mathbb{Z}$ that has the following properties:

    - If $x, y \in \mathfrak{I}$, then $x - y \in \mathfrak{I}$.
    - If $x \in \mathfrak{I}$, then $\forall n \in \mathbb{Z}, xn \in \mathfrak{I}$.

    Further, such an ideal $\mathfrak{I}$ is called a *principal ideal* if there exists a $a \in \mathfrak{I}$ such that $\mathfrak{I} = \{an | n \in \mathbb{Z}\}$. The element a is called the *generator* of the ideal, the notation for the a generator of a principal ideal is $\mathfrak{I} = (a)$.

    Prove that any ideal $\mathfrak{I}$ of $\mathbb{Z}$, with the exception of $\mathfrak{I} = (0)$, is a principal ideal. (In algebra we say that $\mathbb{Z}$ is *Principal Ideal Domain*).

15. Let A, B, and C be pair-wise disjoint, partially ordered sets. If A is order-isomorphic to B, prove that $A \cup C$ is order isomorphic to $B \cup C$.

16. Prove the identity

$$\binom{n+1}{3} = \sum_{k=0}^{n} k(n-k)$$

    using a combinatorial argument.