Sequences and Summation

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1 Notation for generalized summation, union, product and any associative binary operators

We discussed the generalized union and intersection notation previously. Given n sets, $A_1, A_2, \ldots, A_n$, the union and intersection of all those sets can be written as

$$\bigcup_{j=1}^{n} A_j = A_1 \cup A_2 \cup \ldots \cup A_n \tag{1.1}$$

$$\bigcap_{j=1}^{n} A_j = A_1 \cap A_2 \cap \ldots \cap A_n \tag{1.2}$$

Notice the notation: The large union $\bigcup$ is the generalized union symbol. Under the symbol we have the *index* j, as well as the *lower bound*, which in this case is 1, above the symbol we have the *upper bound*, n. The expression in equation 1.1 above should be read as: "Union of sets A_j, with j running from 1 to n". Similarly, the expression in equation 1.2 above should be read as: "Intersection of sets A_j, with j running from 1 to n".

Similarly, we can define generalized summation and product operations. For summation, we use a large capital Greek Sigma letter: $\sum$. For product, we use a large capital Greek Pi letter: $\prod$. For example, with $a < b$ being integers:

$$\sum_{n=a}^{b} n = a + (a+1) + (a+2) + \ldots + (b-1) + b \tag{1.3}$$

$$\prod_{n=a}^{b} n = a \cdot (a+1) \cdot (a+2) \cdot \ldots \cdot (b-1) \cdot b \tag{1.4}$$

As before, the expression in equation 1.3 should be read as: "The sum of n, with n running from a to b"; and the the expression in equation 1.4 should be read as: "The product of n, with n running from a to b". Summation and product are associative operations and the expressions above are well-defined for integers, real numbers, complex numbers, matrices, and many other algebraic structures.

Often we also generalize other associative binary operators (operators that take 2 arguments, like summation: $a + b$), for example the logical AND, $\wedge$ and logical OR, $\vee$:

$$\bigvee_{j=1}^{n} p_j = p_1 \vee p_2 \vee \ldots \vee p_n \tag{1.5}$$

$$\bigwedge_{j=1}^{n} p_j = p_1 \wedge p_2 \wedge \ldots \wedge p_n \tag{1.6}$$

where $p_1, p_2, \ldots, p_n$ are some propositional variables.

Algorithmically, you may consider generalized operators, as described above, as a `for`-loop: The indexing variable and its initial value (the lower bound) are specified under the operator, while the final value (the upper bound), after which the loop terminates is specified above.

Examples Consider a for example:

$$\sum_{n=1}^{5} n = 1 + 2 + 3 + 4 + 5 = 15 \tag{1.7}$$

$$\sum_{k=-3}^{3} k^2 = (-3)^2 + (-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2 = 28 \tag{1.8}$$

$$\prod_{m=2}^{4} (m+1) = 3 \cdot 4 \cdot 5 = 60 \tag{1.9}$$

$$\prod_{j=0}^{2} \frac{1}{2j+1} = \frac{1}{1} \cdot \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15} \tag{1.10}$$

Let's define the sets $A_1 = \{1, 2, -3\}, A_2 = \{2, 5, 6\}, A_3 = \{-3, 2, 3\}$. Then

$$\bigcup_{j=1}^{3} A_j = A_1 \cup A_2 \cup A_3 = \{1, 2, 3, -3, 5, 6\} \tag{1.11}$$

$$\bigcap_{j=1}^{3} A_j = A_1 \cap A_2 \cap A_3 = \{2\} \tag{1.12}$$

$$\bigcap_{m=1}^{2} A_m \cup A_{m+1} = (A_1 \cup A_2) \cap (A_2 \cup A_3) = \{-3, 2, 5, 6\} \tag{1.13}$$

When the operator is commutative (in addition to associative), which is the case for all the operators we have considered above, we can relax the notation further. For example:

$$\sum_{\substack{n \text{ odd integer,} \\ 1 \leq n \leq 5}} n^2 = 1^2 + 3^2 + 5^2 = 35 \tag{1.14}$$

notice that we simply state what we sum over, and do not explicitly specify a range, therefore the upper limit above the operator is omitted. The above should be read as: "Sum of n^2, where n are the odd integers greater or equal to 1 and less or equal to 5". Also notice that with the relaxed notation the order of operations is not explicitly specified, which is why we require commutative operators for the expression to be well-defined. With the sets A_1, A_2, A_3 as above, we can also write:

$$\sum_{n \in A_1} n = 1 + 2 + (-3) = 0 \tag{1.15}$$

$$\prod_{a \in A_3 \cap \mathbb{Z}^+} \frac{a^2}{a+1} = \frac{2^2}{2+1} \cdot \frac{3^2}{3+1} = 3 \tag{1.16}$$

Consider the interval notation $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$. Then:

$$\bigcup_{n=1}^{5} (n, n+1] = (1, 6] \tag{1.17}$$

$$\bigcup_{z \in \mathbb{Z}^+} (n-1, n] = \mathbb{R}^+ \tag{1.18}$$

2 Sequences

A *sequence* is an ordered list of elements, usually numbers, where repetition is allowed. For example: $1, 2, 3, 4, 5$ is a sequence of integers in increasing order from 1 to 5. $1, 3, 2, 4, 5$ is also a sequence of integers from 1 to 5, however the order is different and therefore it is a different sequence. Sequences can be infinite as well, for example: $1, 2, 3, 4, 5, 6, \ldots$, which is the sequence of all positive integers in increasing order.

To formally define a sequence we use an indexed variable, for example a_n, where n is the index. The index is either a positive integer or a non-negative integer. For example: $a_n = n$ with the index ranging over the positive integers till 5, is the same sequence as above:

$$a_1, a_2, a_3, a_4, a_5 \qquad \text{is} \qquad 1, 2, 3, 4, 5 \tag{2.1}$$

Similarly, if $a_n = n$ but n ranges over all the non-negative integers (without an upper bound, that is this is an infinite sequence), then the sequence is $0, 1, 2, 3, 4, 5, 6, 7, \ldots$.

Notice that the definition above is a function in disguise. The domain is a *consecutive* subset of the positive or non-negative integers. The co-domain is generally a subset of the real numbers. If the domain is I, then the sequence a_n is the function $a_n : I \to \mathbb{R}$ that assigns to every index $n \in I$ a value. For example: Let the index set be the integers from 0 to 5, then the following the sequences (on the left we formally define the sequence, and on the right we write out the sequence explicitly):

$$a_n = 2n : \qquad 0, 2, 4, 6, 8, 10 \tag{2.2}$$

$$b_n = n + 1 : \qquad 1, 2, 3, 4, 5, 6 \tag{2.3}$$

$$c_k = \frac{1}{k+1} : \qquad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \tag{2.4}$$

Similarly, if the index set are all the positive integers, then:

$$d_m = 2^m : \qquad 2, 4, 8, 16, 32, 64, 128, 256, 512, \ldots \tag{2.5}$$

$$e_n = \begin{cases} -n & n \text{ is even} \\ \frac{n}{2} & \text{otherwise} \end{cases} : \qquad \frac{1}{2}, -2, 1\frac{1}{2}, -4, 2\frac{1}{2}, -6, 3\frac{1}{2}, -8, 4\frac{1}{2}, \ldots \tag{2.6}$$

To define a sequence we need to be clear about the index range: Do we start from 0 or from 1? Is it a finite sequence? If so, what is the upper bound? Then, we need to define the sequence function, i.e. the mapping from indices to sequence elements.

We give names to a couple of special sequences. A *geomteric progression* is a finite or infinite sequence of the form

$$a, ar, ar^2, ar^3, ar^4, \ldots \tag{2.7}$$

The sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ is an example of geometric progression with $a = 1$ and $r = \frac{1}{2}$.
An *arithmetic progression* is a finite or infinite sequence of the form

$$a, a + d, a + 2d, a + 3d, a + 4d, \ldots \tag{2.8}$$

The sequence $-5, -2, 1, 4, 7, 10, 13, \ldots$ is an example of an arithmetic progression with $a = -5$ and $d = 3$.

2.1 Recursive definition of sequences

The sequences we have considered so far are defined using a *closed-form* expression $a_n = \ldots$. By "closed-form" we mean that we specify a direct formula that can be used to compute each sequence element directly.

At times, it is easier or makes more sense to define a sequence via a *recursive relation*. That is, the value of a sequence element a_n depends on the values of one or more preceding elements, i.e. a_j with $j < n$. For example the sequence $1, 2, 3, 4, 5$ can be defined as

$$a_1 = 1 \quad \text{and} \quad a_n = a_{n-1} + 1 \tag{2.9}$$

with the index set being the positive integers till 5. Notice that we define the *initial conditions*, $a_1 = 1$, as well as the recurrence relation $a_n = a_{n-1} + 1$. To compute the value of an arbitrary element a_n in the sequence, we need to recursively expand the sequence: To compute a_n we need to know the value of

a_{n-1}, to compute the value of a_{n-1} we need the value of a_{n-2}, etc. until we reach a_1, which is known thanks to the initial conditions.

For example: The *Fibonacci sequence*, f_n, is defined as

$$f_0 = 0, f_1 = 1 \quad \text{and} \quad f_n = f_{n-1} + f_{n-2} \tag{2.10}$$

Notice that the index set now starts with 0 and the initial conditions specify the value of two initial elements, this is required because the recursive relation is now of *depth* 2, that is to compute f_n we need knowledge of f_{n-2} in addition to f_{n-1}. Like any linear recurrence relation, a closed-form expression can be found for the Fibonacci sequence and it bears close relation to the so-called *golden ratio* $g = \frac{1-\sqrt{5}}{2}$:

$$f_n = \frac{g^n - (-g)^{-n}}{\sqrt{5}} \tag{2.11}$$

Study the examples in the book. We might briefly discuss recursive relations and sequences later in the quarter.

3 Summation

Using the notation of generalized summation we studied earlier, the sum of a sequence a_n, with n ranging from 1 to N, is

$$\sum_{n=1}^{N} a_n \tag{3.1}$$

Our final topic for today is computing closed-form expressions for such sums, finite or infinite.

We first discuss a few basic properties of the generalized sum. A sum is commutative and associative, therefore the following always holds:

$$\sum_{n=1}^{N} a_n + b_n = \sum_{n=1}^{N} a_n + \sum_{n=1}^{N} b_n \tag{3.2}$$

Multiplication distributes over summation, i.e. we can take a common multiplicative factor out of the sum, for example $ca + cb = c(a + b)$. Similarly:

$$\sum_{n=1}^{N} ca_n = c \sum_{n=1}^{N} a_n \tag{3.3}$$

Let's start with a simple sum:

$$\sum_{n=1}^{N} 1 = \underbrace{1 + 1 + \ldots + 1}_{N \text{ times}} = N \tag{3.4}$$

Another simple sum is the sum of consecutive integers:

$$\sum_{n=1}^{N} n = 1 + 2 + 3 + \ldots + (N-2) + (N-1) + N \tag{3.5}$$

Notice that if we sum the elements marked with the same colour, their sum is always equal: $N+1 = (N-1)+2 = (N-2)+3 = \ldots = N+1$. Then, when N is even, we have $N/2$ such pair, and the sum becomes $\sum_{n=1}^{N} = \frac{N(N+1)}{2}$. When N is odd, we have $(N-1)/2$ such pairs, and we have an unpaired element in the middle of value $(N+1)/2$, thus the sum is $\sum_{n=1}^{N} = \frac{(N-1)(N+1)}{2} + \frac{N+1}{2} = \frac{N(N+1)}{2}$. We conclude that

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2} \tag{3.6}$$

Lemma 3.1. *The sum of the finite arithmetic sequence has the following closed-form expression:*

$$\sum_{n=0}^{N} a + nd = (N+1)\left(a + d\frac{N}{2}\right) \tag{3.7}$$

Proof.

$$\begin{aligned}\sum_{n=0}^{N} a + nd &= \sum_{n=0}^{N} a + \sum_{n=0}^{N} nd = a\left(\sum_{n=0}^{N} 1\right) + d\left(\sum_{n=0}^{N} n\right)\\ &= a\left(1 + \sum_{n=1}^{N} 1\right) + d\left(0 + \sum_{n=1}^{N} n\right) = a(N+1) + d\frac{N(N+1)}{2}\end{aligned} \tag{3.8}$$

as desired. □

Lemma 3.2. *The sum of the finite geometric sequence has the following closed-form expression:*

$$\sum_{n=0}^{N} ar^n = \begin{cases} (N+1)a & \text{if} \quad r = 1 \\ a\frac{r^{N+1}-1}{r-1} & \text{otherwise} \end{cases} \tag{3.9}$$

Proof. If $r = 1$, then

$$\sum_{n=0}^{N} ar^n = \sum_{n=0}^{N} a = (N+1)a \tag{3.10}$$

Otherwise, denote $S = \sum_{n=0}^{N} ar^n$, then:

$$\begin{aligned}rS &= \sum_{n=0}^{N} ar^{n+1} = \sum_{n=1}^{N+1} ar^n = \left(\sum_{n=0}^{N} ar^n\right) - ar^0 + ar^{N+1}\\ &= S + a\big(r^{N+1} - 1\big)\end{aligned} \tag{3.11}$$

that is, $(r-1)S = a(r^{N+1}-1)$. And then,

$$S = a\frac{r^{N+1}-1}{r-1} \tag{3.12}$$

which completes the proof. □

Definition 3.1. An important sum is the *telescopic*-sum, written as follows

$$\sum_{n=1}^{N}(a_n - a_{n-1}) \tag{3.13}$$

Lemma 3.3. *The telescopic sum admits the following closed-form solution:*

$$\sum_{n=1}^{N}(a_n - a_{n-1}) = a_N - a_0 \tag{3.14}$$

Proof.

$$\begin{aligned}&\sum_{n=1}^{N}(a_n - a_{n-1}) \\ &\quad = a_1 - a_0 + a_2 - a_1 + a_3 - a_2 + \ldots + a_{N-1} - a_{N-2} + a_N - a_{N-1}\end{aligned} \tag{3.15}$$

observe that the similarly coloured terms cancel out, that is all the terms a_n with $n \neq 0$ or $n \neq N$. Therefore only $a_N - a_0$ is left on the right-hand side. □

We will conclude with stating that the infinite sum $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ converges and equals 1, without proof.

Lemma 3.4.

$$\sum_{n=1}^{\infty}\frac{1}{2^n} = 1 \tag{3.16}$$