CS40 Winter 2021 Final

March 15, 2021

Notes

- Max grade: 100%.
- **Clearly and fully justify your answers!**. For combinatorial questions you may leave your answers in analytic form as fractions, factorials and binomials/multinomials.
- **You may only use the book (chapters we covered), lectures, lecture notes, homework assignments and their solutions**. No other resources. Individual work only. *Always clearly refer to any result you are using for your work.*
- Please re-read the "Conduct" section in the class syllabus.
- No late submissions! Turn-in what you have by the deadline.

1. Answer the following:

[4%] (a) How many strings of 5 letters are there in which exactly two are vowels (over the 26-letter english alphabet $\{a, b, \ldots, z\}$, out of which 5 are vowels.)?

[4%] (b) A *palindrome* is a word that reads the same either from left or from right (e.g., "rotator"). How many 11-letter palindromes can be arranged over the alphabet of the English letters $\{a, b, \ldots, z\}$ (The words do not have to meaningful)?

[8%] (c) A pawn starts on the lower left corner square of an $n \times n$ chess board ($n \geq 1$). At each step, it can only move a single square to the right or a single square up. In how many different paths can it reach the upper right corner square?

2. Prove:

[6%] (a) If the decimal expansion of a positive integer n is $n = a_k \ldots a_2 a_1 a_0$, where the a_j-s are the decimal digits, then $n \equiv a_k + \ldots + a_1 + a_0 \pmod 9$.

[6%] (b) For $n \geq 0$,

$$\sum_{k=0}^{n} k3^k = \frac{3}{4}[(2n-1)3^n + 1]$$

[8%] (c) For $n \geq 0$,

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

use a combinatorial argument for your proof.
[Hint: Consider counting ordered pairs (x, A) with $x \in A$.]

3. Prove or disprove:

[4%] (a) A permutation of $[n] = \{1, 2, \ldots, n\}$ can be described via a bijection $f : [n] \to [n]$, such that $f(i)$ is the element in position i in the permutation. Then, $f^2 = f \circ f$ is also a permutation.
(For example, $f(1) = 2, f(2) = 4, f(3) = 3, f(4) = 1$ describes the permutation 2431 of [4].)

[5%] (b) $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$ if and only if $A \subseteq B \vee B \subseteq A$.
($\mathcal{P}$ denotes the powerset)

(c) Let $\preceq$ be a total order over $\mathbb{Z}$.
Then, $R = \{((x_1, y_1), (x_2, y_2)) \mid x_1 \preceq x_2 \wedge y_1 \preceq y_2\}$ is a

[3%] i. partial order over $\mathbb{Z}^2$

[3%] ii. total order over $\mathbb{Z}^2$

[8%] (d) Let p be a prime. Then, $p! + 1$ is a prime.

[8%] (e) The set of all functions from $\{a, b, c\}$ to $\mathbb{N}$ is uncountable.

4. Let $p_1, p_2, \ldots, p_n$ be $n > 0$ propositional variables. We define S, the *set of propositions generated* by these propositional variables, recursively as follows:

Basis: $p_1, p_2, \ldots, p_n \in S$.
Recursive: Given $q, r \in S$, then $(q), q \wedge r, q \vee r, \neg q \in S$.

[5%] (a) Prove that every proposition $q \in S$ is well-formed. That is, given some assignment of T/F values to the variables $p_1, \ldots, p_n$, either $q = T$ or $q = F$ (and not both).

(b) The logical equivalence relation $\equiv$ is defined on S as follows:
$\forall q, r \in S$, $q \equiv r$ iff q and r take the same truth value for all possible values of $p_1, \ldots, p_n$.

[5%] i. Prove that it is an equivalence relation.

[9%] ii. Exactly into how many equivalence classes does $\equiv$ partition S? Explain.

(c) We also define the relation $\vdash$ on S:
$\forall q, r \in S$, $q \vdash r$ iff r is a *logical consequence* of q, i.e. $q \rightarrow r$ is a tautology.

[5%] i. Does $\vdash$ define a partial order on S?

[9%] ii. What is the greatest possible integer m, such that $\exists q_1, q_2, \ldots, q_m \in S$ with $q_i \not\equiv q_j$ (for $i \neq j$) and the following holds:
$q_1 \vdash q_2 \vdash \ldots \vdash q_m$,
i.e., there exist propositions $q_1, \ldots, q_m$ that are pair-wise logically inequivalent and form a chain with respect to the relation $\vdash$? Explain your reasoning well.