CS40 Winter 2021 — Lecture Notes
Cardinality, Countability and Infinity

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1 Cardinality of Sets

So far we have discussed finite sets and saw a few examples of common infinite sets ($\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, etc.). In computation and otherwise we are at times concerned with the size, i.e. the *cardinality*, of different sets. For a finite set A we have defined the cardinality of the set $|A|$ to be the count of elements contained in that set. We will now discuss infinite sets and their cardinalities, in more detail.

Definition 1.1. We denote $I_n = \{0, 1, 2, 3, \dots, n-1\} \subset \mathbb{N}$ as the index set consisting of the first n consecutive natural numbers.

How do we "count" the number of elements in infinite sets?

Imagine the following: you are a sheep shepherd in prehistoric times with extremely limited notion of counting. How would you keep track of your sheep? You could take a collection of stones, put a stone on each sheep, and once every sheep has a stone on it you throw out the rest of the stones, and collect all the stones from the sheep. You now have as many stones as you have sheep. The next day, you take your sheep out to graze grass and when you come back you want to make sure that you have the same count of sheep. You do so by placing your stones on the sheep again, if you run out of sheep before you run out of stones, then you know that you have lost some sheep! Notice that no counting, or any knowledge of numbers is needed, we simply found a bijection between the set of stones and the set of sheep.

Let's formalize that process:

Definition 1.2. Given a pair of sets A, B, we say that the A and B have the same cardinality, denoted $|A| = |B|$, iff there exists a bijection from A onto B.

Remark. Sometimes, this relation is described by saying that the sets A and B are *equinumerous*.

Definition 1.3. Given two sets A and B, we define that $|A| \leq |B|$ iff there exists an injection from A into B. Equivalently, $|A| \leq |B|$ iff there exists a surjection from B onto A.

Definition 1.4. We say $|A| > |B|$ iff not $|A| \leq |B|$. That is, $|A| > |B|$ iff there does not exists an injection from A into B.
Equivalently, $|A| > |B|$ iff exists an injection, but not a bijection, from B to A.
Equivalently, $|A| > |B|$ iff exists a surjection from A onto B but no surjection exists from B onto A.

The above definitions imply that for the cardinality of sets an ordering relation can be defined on the equivalence classes of cardinalities. We will come back to that later.

Corollary 1.1. *If $A \subseteq B$, then there exists an injection from A into B and thus $|A| \leq |B|$.*

Previously, we have defined finite and infinite sets, and now we will give equivalent definitions.

Definition 1.5 (Finiteness and infiniteness)**.** A non-empty set A is *finite* iff there exists a bijection from I_n, for some $n \in \mathbb{Z}^+$, onto A. Then, we write $|A| = n$.

Note. This is equivalent to saying that the finite set A has n elements.

Remark. A set that is not finite, is said to be *infinite*.

Definition 1.6 (Cardinality of the empty set)**.** We define $|\emptyset| = 0$ and for every non-empty set A, $|\emptyset| < |A|$.

An immediate consequence is our previous definition of infinity:

Corollary 1.2. *If a non-empty set A is infinite, then $|A| > |B|$ for every non-empty finite set B.*

Proof. For every non-empty finite set B there exists $n \in \mathbb{Z}^+$ such that $|B| = |I_n|$. We want to prove that $\forall n \in \mathbb{Z}^+$, $|A| > |I_n|$, that is there does not exist an injection from A into I_n.
Proof by contradiction: Assume that $\exists m \in \mathbb{Z}^+$ such that there is an injection $\varphi : A \to I_m$. By definition, $\varphi(A) = C \subseteq I_m$ and we can define $\hat{\varphi}(x) = \varphi(x)$ a bijection from A onto C. I_m has m elements, and B is a subset therefore it must have $k \leq m$ elements, therefore exists bijection $\vartheta : I_k \to B$. Then, $\vartheta^{-1} \circ \hat{\varphi} : A \to I_k$ is a bijection as a composition of bijections, which contradicts definition 1.5 and the infinity of A.
Therefore such an m does not exist. □

On the homework, you will prove an equivalent but far more elegant definition of infinity, which I will give here without proof:

Lemma 1.1. *A set A is infinite iff there exists a bijection from A into a proper subset of itself.*

The following immediately follow:

Corollary 1.3. $\mathbb{N}$ *is infinite.*

Proof. $f(n) = 2n$ is a bijection from $\mathbb{N}$ into a proper subset of itself. □

2 Countability

Definition 2.1. A set A is said to be *countably infinite* iff $|A| = |\mathbb{N}|$.

Note. We write $|A| = \aleph_0$.

Remark. We say A is countable, if $|A| \leq |\mathbb{N}|$.

From the definition we conclude that a set A is countable iff there is an injection from A into $\mathbb{N}$.

Countability will play a major role in this week's discussion. Let's start with an important observation:

Lemma 2.1. *Every infinite subset $X \subseteq \mathbb{N}$ is also countably infinite.*

Proof. Without loss of generality, assume that $X \subset \mathbb{N}$. Define $f : \mathbb{N} \to X$ as follows:
X is a subset of the natural numbers, therefore we can find the smallest element in X. We assign $f(0)$ to that smallest integer in X, then we throw that element out. Next, we find the new smallest element in X and assign $f(1)$ to that element, and so on. (this is not a complete proof! The "so on" will be formalized when we talk about mathematical induction)
Clearly, f is a bijection from $\mathbb{N}$ to X, therefore X is countably infinite. $\square$

Theorem 2.2 (There does not exist an infinite cardinality class smaller than countable)**.** *Given a countably infinite set A. Any $B \subseteq A$ is finite or countably infinite.*

Proof. A is countably infinite, therefore $\exists \varphi : \mathbb{N} \to A$ a bijection. Then, $\varphi^{-1}(B) = C \subseteq \mathbb{N}$ and φ^{-1} defines a bijection from B to C. If C is infinite, then by using Lemma 2.1 we deduce that C is countably infinite and thus $|B| = \aleph_0$. Otherwise, C is finite and then B is finite as well. $\square$

The previous theorem implies that countable sets are the *smallest* infinite sets.

Corollary 2.1. *The set of positive integers $\mathbb{Z}^+$ is countably infinite.*

Proof. To see that $\mathbb{Z}^+$ is countable, we notice that $f(z) = z$ is an injection from $\mathbb{Z}^+$ into $\mathbb{N}$. It is easy to see that $\mathbb{Z}^+$ is infinite by noticing that $f(z) = 2z$ is a bijection from $\mathbb{Z}^+$ into a proper subset of itself. $\square$

Corollary 2.2. *$\mathbb{Z}$ is countably infinite.*

Proof. Define the following function $f : \mathbb{N} \to \mathbb{Z}$,

$$f(n) = \begin{cases} \frac{n}{2} & \text{if} \quad n \text{ even} \\ -\frac{n+1}{2} & \text{if} \quad n \text{ odd} \end{cases} \tag{2.1}$$

f is a bijection (prove it!), therefore $|\mathbb{N}| = |\mathbb{Z}| = \aleph_0$. $\square$

The Hilbert Grand Hotel The "Hilbert Grand Hotel" is a funny construct. Hilbert's hotel has a countably infinite collection of rooms, numbered $1, 2, 3, \ldots$ and interesting paradoxes arise that demonstrate why infinite sets, in terms of their "size"—cardinality—do not behave like finite sets.

Example 2.1. Assume that Hilbert's hotel is full. A guest arrives at the hotel, can we find them a room?

Solution. Sure. Take the guest at room number 1 and move them to room number 2, take the displaced former guest of room number 2 and move them to room number 3, and so on. We have moved all of our guests, and found them all a new room, however room 1 is now vacant and we can assign our new guest to that room.

We have effectively shown that $\{1, 2, 3, \ldots\}$ is bijective with $\{2, 3, 4, \ldots\}$ via the bijection $\varphi(a) = a + 1$. ■

Example 2.2. A bus with $n > 0$ seats arrives full with guests at the hotel. Can we find all of them a room?

Solution. Similar to before, we take the guest at room number 1 and move them to room number $n + 1$, we take the displaced guest from room $n + 1$ and move them to room $2n + 1$, etc. Then we take the guest at room number 2 and move them to room number $n + 2$, we take displaced guest from room $n + 2$ and move them to $2n + 2$, etc. We repeat for all the guest in rooms 1 through n, once we are done those rooms are vacant and we can assign our n new guests to those rooms.

We have shown that $\mathbb{Z}^+$ is bijective with $\{n + 1, n + 2, \ldots\}$ for any $n \in \mathbb{Z}^+$ by defining the bijection $\varphi(a) = a + n$. ■

Example 2.3. A bus with countably infinite seats, full with guests arrives at the hotel. Can we find all of them a room?

Solution. Yes. Let those guests be $\{g_1, g_2, g_3, \ldots\}$. We take all of our current guests, and reassign them rooms based on the rule $n \to 2n$. That is, guest in room 1 moves to room 2, guest in room 2 moves to room 4, the guest from 4 moves to 8, etc. Likewise the guest from room 3 moves to 6, guest in room 6 moves to 12, etc. Once we are done, all of our odd rooms are vacant, and we assign rooms to our new guests as follows: $g_j \to 2j - 1$.

We have effectively shown that $\mathbb{Z}^+$ is bijective with the even positive integers as well as the odd positive integers. ■

Examples

Example 2.4. Given A, B countably infinite sets, show that $A \cup B$ is countably infinite.

Solution. A and B are countably infinite, therefore there are bijections $f : \mathbb{N} \to A$, $g : \mathbb{N} \to B$. Define $\varphi : \mathbb{N} \to (A \cup B)$ as follows:

$$\varphi(x) = \begin{cases} f(\frac{x}{2}) & \text{if} \quad x \text{ even} \\ g(\frac{x-1}{2}) & \text{if} \quad x \text{ odd} \end{cases} \tag{2.2}$$

φ is a surjection (but not necessary an injection!) from $\mathbb{N}$ to $A \cup B$, therefore $|A \cup B| \leq |\mathbb{N}|$.

To show that $A \cup B$ is infinite, we notice that $A \subseteq A \cup B$, therefore $|A| \leq |A \cup B|$, therefore $A \cup B$ is infinite.

Therefore, $A \cup B$ must be countably infinite. ■

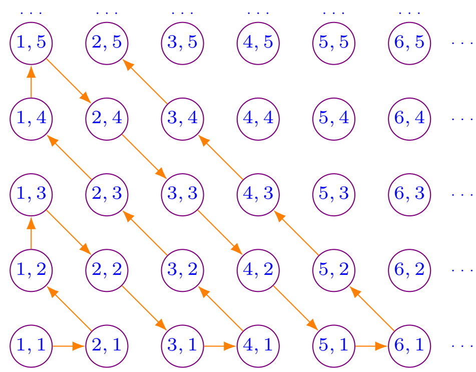
From the example above, we can deduce that $A \cup B$ is countable whenever A, B are (finite or infinite) countable.

Example 2.5. Given $A_1, A_2, \ldots, A_n$ countably infinite sets, show that $\bigcup_{j=1}^{n} A_j$ is countably infinite.

Solution. Same idea. The union is trivially infinite. Let's show that it is countable. There exist $f_j : \mathbb{N} \to A_j$ bijections for all $1 \leq j \leq n$. We define f from $\mathbb{N}$ to the union of all sets as follows:
For the first n non-negative integers we define $f(0) = f_1(0), f(1) = f_2(0), f(2) = f_3(0), \ldots, f(n-1) = f_n(0)$.
For the next set of n integers we define $f(n) = f_1(1), f(n+1) = f_2(1), f(n+2) = f_3(1), \ldots, f(2n-1) = f_n(1)$.
And so forth. In general, for every integer $z = sn + r$, where $s \geq 0$ and $0 \leq r < n$, we define $f(z) = f(sn + r) = f_{r+1}(s)$. In the future, we will see that every integer z has a unique such decomposition. Therefore, f indeed defines a surjection, and the union is countably infinite. ■

Example 2.6. $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

Solution. Take the ordered pairs $(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and write them on a 2-dimensional graph as follows:



The circles are all the elements of $\mathbb{Z}^+ \times \mathbb{Z}^+$, while the orange line describes our *counting strategy*: We to construct a bijection $\varphi : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ by counting. We start at $(1,1)$, that is we assign $\varphi(1,1) = 1$, then $\varphi(2,1) = 2$, then $\varphi(1,2) = 3$, and so forth following the orange line.

Then, notice that the first diagonal has 1 element—$(1,1)$—the second diagonal has two elements—$(2,1)$, $(1,2)$—and so forth. How many elements have we counted after counting k diagonals? $\sum_{n=1}^{k} n = \frac{k(k+1)}{2}$. At which element do we start counting the k^{th} diagonals? If k is even then $(k,1)$, otherwise $(1,k)$. Therefore, given any odd $k \in \mathbb{Z}^+$ and any $1 \le l \le k$:

$$\varphi(l, k-l+1) = \frac{(k-1)k}{2} + l \tag{2.3}$$

and given any even $k \in \mathbb{Z}^+$ and any $1 \le l \le k$:

$$\varphi(k-l+1, l) = \frac{(k-1)k}{2} + l \tag{2.4}$$

φ is a bijection by construction, and therefore $|\mathbb{Z}^+ \times \mathbb{Z}^+| = \aleph_0$. ■

Corollary 2.3. *A countably infinite union of countably infinite sets is countable.*

Proof. Let $A_1, A_2, \ldots$ be the countably infinite collection of sets. We want to show that $U = \bigcup_{j=1}^{\infty} A_j$ is countable.

Let $f_j : \mathbb{Z}^+ \to A_j$ bijections. Define the function $\varphi : \mathbb{Z}^+ \times \mathbb{Z}^+ \to U$ as follows: $\varphi(i,j) = f_j(i)$. φ is clearly surjective (prove it!), therefore $|U| \le |\mathbb{Z}^+ \times \mathbb{Z}^+| = \aleph_0$. The union is trivially infinite, therefore $|U| = \aleph_0$. □

Highly-surprising is the next results:

Corollary 2.4. $\mathbb{Q}$ *is countable.*

Proof. Let $\varphi : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$ be as follows: Every rational $q \in \mathbb{Q}$ can be written (by definition) as $q = \frac{n}{m}$ with $n, m \in \mathbb{Z}$ and $m \neq 0$. Then, $\varphi(\frac{n}{m}) = (n, m)$ is an injection. $\mathbb{Z}$ is countable, therefore $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ is countable (prove it!), therefore $|\mathbb{Q}| = \aleph_0$. □

Example 2.7. The set of all finite computer programs is countable.

Proof. Let S be the finite set of all valid symbols used in some given programming language. Let a computer program of length $l > 0$ be an ordered sequence of l symbols from S (it might not be syntactically valid, but we don't care). Then, we denote $P_l = S^l = S \times S \times \ldots \times S$ the set of all computer programs of length l. The set of all computer programs of any finite length is then $P = \bigcup_{l=1}^{\infty} P_l$, which is a countably infinite union of finite sets, and therefore is countable. □

3 The Cantor-Schröder-Bernstein Theorem

We present the first fundamental theorem of set cardinalities. I will provide a full proof, which is beautiful, and I strongly encourage studying it. For the proof the following Lemma will be needed:

Lemma 3.1. *If there exists an injection f from A into a proper subset of itself, $B \subset A$, then $|A| = |B|$.*

Proof. Let $X_0 = A \setminus B$ and recursively: $X_{n+1} = f(X_n) = (\underbrace{f \circ f \circ \ldots \circ f}_{n+1 \text{ times}})(X_0)$.

Let $X = \bigcup_{j=0}^{\infty} X_j$ and

$$g(a) = \begin{cases} f(a) & \text{if} \quad a \in X \\ a & \text{otherwise} \end{cases}$$

be a function with domain A. We first show that it is a function to B:
Let $a \in X$, then $f(a) \in B$ by definition of f.
Let $a \notin X$, then $a \notin X_0 = A \setminus B$, however $a \in A$ therefore $a \in B$ and thus $g(a) = a \in B$. Therefore g is a function from A to B.

We now show that it is injective:
Let $s, t \in A$, assume that $g(s) = g(t)$. We want to conclude that $s = t$. There are four cases:

1. $s \in X, t \in X$: Then, $f(s) = f(t)$ and $s = t$ because f is an injection.

2. $s \notin X, t \notin X$: Then, $s = t$ by definition of g.

3,4. Exactly one of s, t is in X, and without loss of generality we assume $s \in X$, $t \notin X$. Then, $\exists n \in \mathbb{N} s \in X_n$, by our construction of $X_{n+1} = f(X_n)$, thus $g(s) = f(s) \in X_{n+1} \subseteq X$. However, $t \notin X \Rightarrow g(t) = t \notin X$, contradicting $g(s) = g(t)$.

thus g is injective.

Finally, we need to show that g is a surjection:
For any $b \in B$, we want to find $a \in A$ such that $g(a) = b$. If $b \notin X$, then $a = b$. Otherwise, $b \in X$. $b \in B$, therefore $b \notin X_0$, therefore $\exists n \in \mathbb{N}, b \in X_{n+1}$. Then, $b \in f(X_n)$ and thus $\exists a \in X_n \subseteq A \Rightarrow f(a) = b$, $a \in X \Rightarrow g(a) = f(a) = b$.

Therefore, g is a bijection from A to B, and we conclude $|A| = |B|$. □

We are now ready to state the theorem and prove it.

Theorem 3.2 (Cantor-Schröder-Bernstein Theorem)**.** *$|A| \leq |B|$ and $|B| \leq |A|$ iff $|A| = |B|$.*

Proof. One direction is trivial. If $|A| = |B|$ then exists a bijection $\varphi : A \to B$. φ is an injection therefore $|A| \leq |B|$, however $\varphi^{-1} : B \to A$ is also an injection therefore $|B| \leq |A|$.

The other direction is more difficult. We are given $|A| \leq |B|$ and $|B| \leq |A|$ and we want to deduce $|A| = |B|$. The premises imply that there exist injections $f : A \to B$ and $g : B \to A$, and we assume that f, g are both not surjective, as otherwise we are done.

$g \circ f$ defines an injection (as a composition of injections) from A into $g(B) \subset A$, therefore by Lemma 3.1, $|g(B)| = |A|$ and there exists a bijection $h : A \to g(B)$. However $\hat{g} : B \to g(B)$ is a bijection defined by $\hat{g}(x) = g(x)$, therefore $\hat{g}^{-1} \circ h : A \to B$ a bijection (as a composition of bijections).
Therefore $|A| = |B|$. □

At times, it might be difficult to find an explicit bijection between sets to show that they have the same cardinality. However, finding a pair of injections, from one set into the other and vice versa, is generally easier. The Cantor-Schröder-Bernstein Theorem states that the two things are equivalent.

4 Equivalence Classes of Cardinality

We will now show that there are sets which are strictly "larger"—cardinality-wise—than countably infinite sets, e.g., the set of real numbers, $\mathbb{R}$. That is, the set of real numbers is infinite and yet no injection from $\mathbb{R}$ into $\mathbb{N}$ exists. Furthermore, we will show that there is no upper bound on the "size" of a set, i.e. for every set we can immediately construct another set with a strictly greater cardinality.

This is formalized in the following theorem, and the proof is simple, beautiful and should remind you of Russell's paradox.

Theorem 4.1 (Cantor's diagonalization)**.** *Let X be a non-empty set, then $|X| < |\mathcal{P}(X)|$.*

Proof. Proof by contradiction. Assume that there exists a non-empty X as well as a bijection $\sigma : X \to \mathcal{P}(X)$. Then, denote $Y = \{x \in X : x \notin \sigma(x)\}$. $Y \subseteq X \Rightarrow Y \in \mathcal{P}(X)$, therefore $\exists y \in X$ such that $\sigma(y) = Y$.
There are a couple of options, $y \in Y$ or $y \notin Y$, but both result in an immediate contradiction (show it!). And we conclude that $|X| < |\mathcal{P}(X)|$. □

Lemma 4.2. *If $|A| = |B|$, then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.*

Proof. Given a bijection $f : A \to B$, it also defines a bijection $\mathcal{P}(A) \to \mathcal{P}(B)$ when we consider its images (show it!). □

We present a short, but useful, corollary:

Corollary 4.1. *If $|A| \geq \aleph_0$, then there exists $B \subseteq A$ such that $|B| = \aleph_0$.*

Proof. $|A| \geq \aleph_0$ therefore there exists an injection φ from $\mathbb{N}$ into A. $B = \varphi(\mathbb{N})$ is a set as required. □

We now turn out attention to the real numbers.

Lemma 4.3. *Let $X = (0, 1)$ (the interval of real numbers greater than 0 and less than 1), then $|X| = |\mathbb{R}|$.*

Proof. We construct an explicit bijection $f : (0, 1) \to \mathbb{R}$, as follows:
Define $g(x) = 2x - 1$ a bijection from $(0, 1)$ to $(-1, 1)$ (prove it!). Also define $h(x)$ as follows:

$$h(x) = \begin{cases} \frac{1}{x} - 1 & \text{if} \quad x > 0 \\ \frac{1}{x} + 1 & \text{if} \quad x < 0 \\ 0 & \text{otherwise} \end{cases}$$

which is a bijection $(-1, 1) \to \mathbb{R}$ (prove it!).
Then, $f = h \circ g$ is a bijection from $(0, 1)$ to $\mathbb{R}$ and $|X| = |\mathbb{R}|$. □

Corollary 4.2. $|[0, 1)| = |(0, 1)|$.

Proof. $f(x) = x$ injects $(0, 1)$ into $[0, 1)$, and $g(x) = \frac{x+1}{2}$ injects $[0, 1)$ into $(0, 1)$ (verify!), therefore by the Cantor-Schröder-Bernstein Theorem $|[0, 1)| = |(0, 1)|$. □

Corollary 4.3. $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$*, and we denote* $|\mathbb{R}| = \mathfrak{c}$.

Proof. Consider the following injection from $\mathcal{P}(\mathbb{Z}^+)$ into $[0, 1)$:
Given $S \subseteq \mathbb{Z}^+$, $f(S) = \sum_{n \in S} 10^{-n}$.

Likewise, consider the following injection from $[0, 1)$ into $\mathcal{P}(\mathbb{Q})$:
$g(x) = \{q \in \mathbb{Q} : q \leq x\}$ (by the density of the rational numbers).

We have shown that $|[0, 1)| \leq |\mathcal{P}(\mathbb{Q})| = |\mathcal{P}(\mathbb{N})|$ and that $|\mathcal{P}(\mathbb{N})| = \mathcal{P}(\mathbb{Z}^+) \leq |[0, 1)|$. Therefore $|\mathcal{P}(\mathbb{N})| = |[0, 1)| = |\mathbb{R}|$. □

We call the cardinality class $\mathfrak{c}$ "continuum". We have now shown that the real numbers (or in practice, any continuous subset of the real numbers) is uncountably infinite, that is the set of real numbers is strictly greater than the set of the natural numbers. Can we find even greater sets? Of course, by applying Theorem 4.1: $|\mathcal{P}(\mathbb{R})| > |\mathbb{R}|$. And we will formalize those equivalence classes in a moment.

Also of interest is the question, is there a class between countably infinite and continuum? That is, is there a set A such that $\aleph_0 < |A| < \mathfrak{c}$? We assume that the answer is no, and that is known as the *Continuum hypothesis*:

Theorem 4.4 (The continuum hypothesis)**.** *"There is no set whose cardinality is strictly between that of the integers and the real numbers."*

We do not provide a proof, moreover it has been shown that under ZFC, the most common axiomatic foundation of Discrete Mathematics, the question above can not be formally proven.

Before we continue, notice that we can deduce the following interesting result

Corollary 4.4. *On the homework you will show that the set of all functions $f : \mathbb{Z}^+ \to \{0, 1, 2\}$ is not countable, however we have shown that all finite computer programs are countable. Therefore, for almost all such functions, we can not write a computer program to output them.*

Or, equivalently, almost nothing is computable! We will talk about that next week a little more.

How many equivalence classes are there then? We are now ready to start classifying equivalence classes of infinite sets.

Definition 4.1. Let $\beth_0 = \aleph_0$ and define $\beth_{n+1} = 2^{\beth_n}$.

Remark. That is, $|\mathbb{N}| = \beth_0$, $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = \beth_1$, $|\mathcal{P}(\mathbb{R})| = \beth_2$, etc.

What cardinality classes have we found so far? The countably infinite collection of finite classes $(0, 1, 2, 3, \ldots)$, as well as the countably infinite collection of infinite classes $\beth_0, \beth_1, \beth_2, \ldots$. Can we find more equivalence classes? That is, can we construct a set that does not fall into any of the equivalence classes described so far? Yes!

Definition 4.2. Define the class $\beth_\omega = \bigcup\limits_{n \in \mathbb{N}} \beth_n$.

Remark. That is we construct a set A_ω, and define $|A_\omega| = \beth_\omega$, as follows:

$$A_\omega = \bigcup_{n=0}^{\infty} \underbrace{\mathcal{P}(\mathcal{P}(\ldots \mathcal{P}}_{n\text{-times}}(\mathbb{N}) \ldots)) = \mathbb{N} \cup \mathcal{P}(\mathbb{N}) \cup \mathcal{P}(\mathcal{P}(\mathbb{N})) \cup \ldots$$

Corollary 4.5. *$\beth_\omega$ is not equivalent to any $\beth_n$.*

Proof. Assume that there exists an $n \in \mathbb{N}$ such that $|A_\omega| = \beth_n$. However there is a set $B \subseteq A_\omega$ such that $|B| = \beth_{n+1}$ and by Theorem 4.1 $\beth_{n+1} > \beth_n$, contradicting $|A_\omega| = \beth_n$. □

We can go on:

Definition 4.3. $\beth_{\omega+1} = 2^{\beth_\omega}$, that is $|\mathcal{P}(A_\omega)| = \beth_{\omega+1}$, and so on for every $n \in \mathbb{N}$ we define $\beth_{\omega+n}$.

Repeating the same process, we can keep taking the infinite union of all the equivalence classes we have defined up to a point, in-order to keep creating new equivalence classes ad infinitum! And we end-up with the following conclusion, one of the most amazing results in Mathematics:

Theorem 4.5. *There does not exist a set of equivalence classes C.*

Proof. Assume that such C exists. Then, in a manner similar to our previous discussion we can construct $c_{\text{bigger}} = \bigcup\limits_{c \in C} c$, and it must hold that $c_{\text{bigger}} \notin C$. □