CS40 Winter 2021 Homework #9

March 1, 2021

Notes

- You may work with a partner in order to understand the problems and discuss how to approach them. If you do so, write clearly on your assignment the name of the student you collaborated with.
- *Justify your answers! A numeric answer is not sufficient. You may leave your answers in analytic form as fractions, factorials and binomials/multinomials.*
- Please re-read the "Conduct" section in the class syllabus.
- No late submissions! Turn-in what you have by the deadline.

. . . .

1. (a) How many three-digit natural numbers can be assembled from the digits $2, 3, 5, 6, 7$, if the digits are not repeated? How many of these numbers are divisible by five?

   (b) If the number of elements n is increased by two, then the number of 3-combinations from those n elements increases by 324. Find the number of elements n (after increase).

   (c) How many positive integers divisible by five smaller than 8000 exist, if they are only made out of the digits $0, 1, 2, 5, 7, 9$?

   (d) How many permutations of the 26 letters of the English alphabet do not contain any of the strings "fish", "bird" or "sock"?

   (e) We have $6\,\mathrm{l}$ of milk and 4 empty bottles of milk. Using a $0.5\,\mathrm{l}$ measuring cup, in how many ways can we distribute our milk into the bottles? (you may assume distinguishable bottles).

   (f) Assume that the relation "friend" is symmetric. Show that if $n \geq 2$, then in any group of n people there are two with the same number of friends in that group.

2. Suppose there are 15 blue balls and 15 black balls, each marked with an integer between 1 and 100 (inclusive), and no integer appears on more than one ball (of any colour).

   (a) How many possible ways are there to draw 10 balls and place them in a line.

(b) How many possible ways are there to draw 10 balls and put them in a box (i.e. ignoring order).

(c) The value of a pair of balls is the sum of the numbers on the balls. Show there are at least two pairs, consisting of one blue and one black ball, with the same value. Show that this is not necessarily true if there are 13 balls of each colour. (Be careful! Showing that a particular argument fails is not enough to show that something can not be done in general.)

3. (a) i. How many 9-bit binary strings are there which begin with 101?

ii. How many of the strings you counted in part (a) also end with 111?

iii. How many 9-bit binary strings are there which begin with 101 or end with 111?

(b) How many strings of four decimal digits

i. do not contain the same digit twice?

ii. end with an even digit?

iii. have exactly three digits that are 9s?

(c) A test with 20 questions is a multiple choice test with 5 answers for each question but only 1 correct answer to each question. How many ways are there to have

i. no correct answers?

ii. exactly 6 correct answers?

iii. at least 6 correct answers?

4. Consider the equation $x_1 + x_2 + x_3 + x_4 = 22$. Calculate the number of solution if:

(a) x_j-s are non-negative integers.

(b) x_j-s are non-negative integers and $1 < x_1 < 7$, $3 \leq x_2 \leq 5$, $x_3 \leq 7$, $1 < x_4 \leq 13$.

5. (a) Let $n \geq k$ positive integers, prove

$$\binom{n}{k} \leq \frac{n^k}{2^{k-1}}$$

(b) Let $n \geq k$ positive integers, prove

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

(c) For any natural numbers n, k, prove the diagonal sum property:

$$\sum_{r=0}^{n} \binom{r+k}{r} = \binom{k+n+1}{n}$$

The diagonals are the elements coloured in Figure 1 in the lecture notes (Pascal's triangle), and we claim that the sum of the first $n+1$

elements of the diagonal is simply the element directly under and to the right of the last element in the diagonal.

[Hint: You may want to use the "column sum property", theorem 4 in the book, chapter 6.4.]

(d) Given a natural number n, show that

$$\sum_{k=0}^{n} \binom{n-k}{k} = f_{n+1}$$

where f_{n+1} is the $n+1$ Fibonacci number.

6. (a) Let $n \in \mathbb{Z}^+$. Is it true that $(4n)!$ is always divisible by $(n!)^4$? Justify!

   (b) Calculate the sum of all fractions of the form $\frac{1}{k!r!}$ where k, r are all the possible pairs of non-negative integers whose sum is 18.

7. Prove the following version of the inclusion-exclusion principle:

$$\left| \bigcup_{j=1}^{n} S_j \right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{I \in \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]} \left| \bigcap_{m \in I} S_m \right|$$

where $S_1, S_2, \ldots, S_n \subseteq U$ are sets with a universe U, with $n > 0$.