CS40 Winter 2021 Homework #8

February 23, 2021

Notes

- You may work with a partner in order to understand the problems and discuss how to approach them. If you do so, write clearly on your assignment the name of the student you collaborated with.
- **Justify your answers!**
- Please re-read the "Conduct" section in the class syllabus.
- No late submissions! Turn-in what you have by the deadline.

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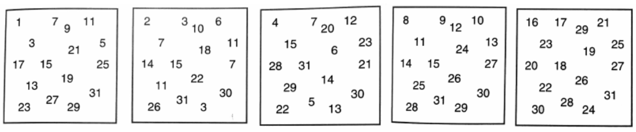
1. (a) Show that $3|k^3 - k$ for any integer k.

   (b) Show that there are no solutions to the equation $p^2+q^2 = r^2+s^2+t^2$ where p, q, r, s, t are primes. [Hint: Consider congruence modulo 8.]

2. Choose any number you see on any of the cards. For each of the five cards: if your number is on the card, circle the number on the upper left corner of that card. The sum of these circled numbers is your chosen number.



   (a) Explain why this works.

   (b) Describe a similar trick based on the number 3.

3. Let R be a relation on a set A. We denote $R^2 = R \circ R$ as the relation composition, and generalize to $R^n = \underbrace{R \circ R \circ \ldots \circ R}_{n\text{-times}}$.

   (a) Prove, using induction, that if R is transitive then $R^n \subseteq R$.

   (b) Prove that if $R^2 \subseteq R$, then R is transitive.

4. Give recursive definition of each of these sets of ordered pairs of positive integers.

   (a) $S_1 = \{(a, b) \mid a, b \in \mathbb{Z}^+ \quad \text{and} \quad a + b \text{ is odd}\}$

(b) $S_2 = \{(a,b) \mid a,b \in \mathbb{Z}^+ \quad \text{and} \quad a|b\}$
(c) $S_3 = \{(a,b) \mid a,b \in \mathbb{Z}^+ \quad \text{and} \quad 3|(a+b)\}$

5. We denote f_n as the n-th Fibonacci number.

   (a) Prove the following statement by induction: For $n \geq 1$,

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}\frac{n(n+1)}{2}$$

   (b) Prove by induction that for each integer $n \geq 2$, $f_n < \left(\frac{7}{4}\right)^{n-1}$.
   (c) Prove that $\gcd(f_n, f_{n+1}) = 1$ for every $n \geq 1$.
   (d) Show by induction on n that for $n \geq 0$,

$$\sum_{i=0}^{n} f_i = f_{n+2} - 1 \ .$$

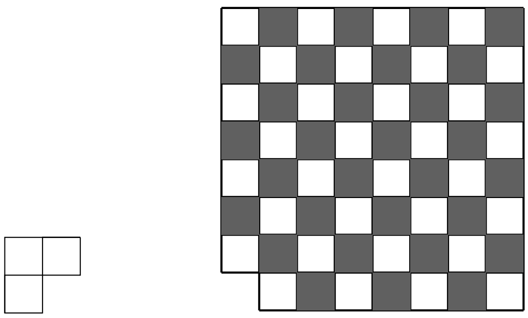
   (e) Prove $(f_n)^2 = f_{n-1}f_{n+1} - (-1)^n$, for $n \in \mathbb{Z}^+$.

6. For a finite set of numbers S, define $\max S$ to be the largest number in the set and $\min S$ the smallest number in the set S. For each of the following relations on $A = \mathcal{P}(\{1,2,3,4\})$, decide if it is reflexive, irreflexive, transitive, or antisymmetric.

   (a) $R_1 = \{(S,T) \in A \times A \mid \max S \leq \max T\}$
   (b) $R_2 = \{(S,T) \in A \times A \mid \max S \leq \min T\}$
   (c) $R_3 = \{(S,T) \in A \times A \mid \max S < \min T\}$

   For each property that a relation fails to satisfy, give a counterexample.

7. Let an m-chessboard be a chessboard with $m \times m$ squares (instead of the usual 8). Consider an 2^n-chessboard with one (arbitrary) corner square removed. Prove, using induction, that for every $n \geq 1$ such a chessboard can be perfectly covered by L-shaped 3-square dominos.



L-shaped domino $\qquad$ 2^n-chessboard ($n = 3$) with a corner removed

8. We need to send a letter using stamps. We have an unlimited supply of 5-cent and 12-cent stamps, but no other stamps. Prove that we can exactly cover the shipping costs of c-cents, for any integer $c > 43$.

9. (a) You are given 3 seemingly identical rings, supposedly made of gold. One of them is fake: it is made of copper and thus weighs less than a gold ring. You have access to a simple balance that you can use to compare the weights of objects. Using only a single weighing (you get to use the balance once) how would you detect the fake ring?

   (b) Given 9 rings (exactly one is fake), how would you detect the fake one if you are allowed to do only two weighings?

   (c) Generalize and prove by induction that it is possible to detect the one fake ring from 3^n rings with only n weighings, $n \geq 0$.