CS40 Winter 2021 Homework #7

February 22, 2021

Notes

- You may work with a partner in order to understand the problems and discuss how to approach them. If you do so, write clearly on your assignment the name of the student you collaborated with.
- Justify your answers!
- Please re-read the "Conduct" section in the class syllabus.
- No late submissions! Turn-in what you have by the deadline.

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1. Prove that there exists infinitely many numbers of the form $a_n = \frac{n(n+1)}{2}$, where n is some positive integer, such that every pair a_n, a_m (for $n \neq m$) are relatively prime.

   [Hint: Assume there exists a finite sequence $a_{n_1} < a_{n_2} < a_{n_3} < \ldots < a_{n_m}$, where n_j are increasing positive integers. Consider $k = \prod a_{n_j}$, i.e. the product of all these integers. Show that using k we can construct a new number that fulfils the requirements.]

2. In this question we construct the famous *Cantor set*:
   We define $C_0 = [0, 1]$.
   Then, we remove the middle fourth from that interval, and end up with $C_1 = C_0 \setminus (\frac{3}{8}, \frac{5}{8}) = [0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$.
   At the next step, we remove the middle fourth from the two intervals we ended up with in C_1 to define C_2. Continuing in this fashion we define C_n for arbitrary $n \in \mathbb{N}$, and C_{n+1} is constructed from C_n by removing the middle fourth from each interval in C_n. Notice that we always remove an open interval.
   Illustration (the black lines are C_n starting at $n = 0$ at the top):

   The Cantor set is then defined to be the countably infinite intersection:

$$\mathcal{C} = \bigcap_{n=0}^{\infty} C_n$$

(a) The length of some interval (c, d) (open, closed or half open) on the real line is defined to be the value $d - c$.
Then, the length of C_0 is clearly 1, the length of C_1 is $(\frac{3}{8} - 0) + (1 - \frac{5}{8}) = \frac{3}{4}$. Each C_n is clearly a union of finite intervals, so it must have a length. Explicitly calculate the length of C_n. What is then the length of $\mathcal{C}$? [Hint: Think sequences and summation.]

(b) Prove that $\mathcal{C}$ is nowhere dense (that is, there does not exists a dense subset of $\mathcal{C}$). [Hint: By contradiction, assume that such a subset exists and consider its length.]

(c) We will now prove that $\mathcal{C}$ is uncountable as follows: Assume, by contradiction, that $\mathcal{C}$ is countable and therefore we can write all of its points as a (possibly infinite) sequence $\mathcal{C} = \{x_1, x_2, x_3, \ldots\}$.
Note that $\mathcal{C} \subset C_1 = [0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$, therefore the point x_1 must be in exactly one of these two disjoint intervals, $[0, \frac{3}{8}], [\frac{5}{8}, 1]$ and denote *the other* interval I_1, i.e. $x_1 \notin I_1$ and $I_1 = [0, \frac{3}{8}]$ or $I_1 = [\frac{5}{8}, 1]$ (but not both).
Complete the proof by constructing a point $x \in \mathcal{C}$ which is not in the list above. Do so by building a countably infinite sequence of *closed, non-empty* intervals $\ldots \subset I_m \subset \ldots \subset I_2 \subset I_1 \subset C_0$, such that $x_m \notin I_m$ for every $m \in \mathbb{Z}^+$. You may assume that the countably infinite intersection of such a sequence of intervals is non-empty (this is known Cantor's intersection theorem).

3. Find a (non-empty) relation on the set $\{a, b, c\}$ that is neither

(a) reflexive nor irreflexive.

(b) symmetric nor antisymmetric.

(c) symmetric nor asymmetric.

4. Suppose that $\sim$ is an equivalence relation on some set A. Prove the following.
(Always, explicitly, specify which properties of the relation you are using!)

(a) Prove that for all $a, b \in A$, the following conditions are equivalent (that is, each is a sufficient condition for the other two).

i $a \sim b$

ii $[a] \cap [b] \neq \emptyset$

iii $[a] = [b]$

(b) Show that the set of all equivalence classes $A^\sim = \{[a] : a \in A\}$ partitions A.

5. Let P be the set of all people. The function $f : P \to \mathbb{N}$ assigns to each person their (integer) age. Let $\sim$ be an equivalence relation on P, such that $a \sim b$ iff (if and only if) a and b are of the same age. We denote the set of all equivalence classes in P, with respect to $\sim$, as $P^\sim = \{[p] : p \in P\}$.

A function $\hat{f} : P^\sim \to f(P)$ is defined as $\hat{f}([p]) = f(p)$. Show that $\hat{f}$ satisfies the definition of a function, explain what $\hat{f}$ represents and prove that it is a bijection.

6. Let $\preceq$ be a well ordering on a set A. Show that it is a total ordering. (Always, explicitly, specify which properties of the relation you are using!)

7. Let G and H be equivalence relations on set A. Show that $G \circ H$ is an equivalence relation on A iff $G \circ H = H \circ G$.

8. Definition: given a set A and a partial order $\preceq$, a subset $C \subseteq A$ is called a *convex subset* when for every pair $a, b \in C$, if $x \in A$ such that $a \preceq x \preceq b$, then $x \in C$.

   Let A and B be partially ordered sets, with respect to orders $\preceq_A$ and $\preceq_B$, respectively. Let $f : A \to B$ be an order preserving function, i.e. $x \preceq_A y$ iff $f(x) \preceq_B f(y)$. Prove or disprove:

   (a) If $C \subseteq A$ is a convex subset, then $f(C)$ is a convex subset.

   (b) If $C \subseteq B$ is a convex subset, then $f^{-1}(C)$ is a convex subset (remember! f^{-1} is the pre-image, f does not have to be invertible).