CS40 Winter 2021 Homework #6

February 10, 2021

Notes

- You may work with a partner in order to understand the problems and discuss how to approach them. If you do so, write clearly on your assignment the name of the student you collaborated with.
- Justify your answers!
- Please re-read the "Conduct" section in the class syllabus.
- No late submissions! Turn-in what you have by the deadline.

Definitions Given a set D and a set of sets $P = \{A_1, A_2, \ldots\}$ (finite or infinite). If following conditions hold

i P is a set of subsets of D: $P \subseteq \mathcal{P}(D)$, where $\mathcal{P}(D)$ is the powerset.

ii The partitioning sets are pair-wise disjoint:
$\forall X, Y \in P \ (X \neq Y \to X \cap Y = \emptyset)$.

iii Every element in D must be in some set of the partition:
$\bigcup_{A \in P} A = A_1 \cup A_2 \cup \ldots = D$.

then P is a called partition of D.
In other words, a partition is a set of subsets such that all the sets are pair-wise disjoint and their union is D.

A partition can also be infinite, in which case we define:

- An infinite partition is a partition with infinitely (countable or uncountable) many sets.
- An infinite partition that consists of infinite sets is a partition with infinitely (countable or uncountable) many sets where each set is also infinite (countable or uncountable).

Note that when the partition is uncountable, we can not index the sets in the partition using integers!

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1. (a) Find $d = \gcd(36, 55)$ by the Euclidean algorithm.
   (b) Use your work in part (a) to find $r, s \in \mathbb{Z}$ such that $d = 36r + 55s$.
   (c) Use your work in part (b) to solve the equation $36x \equiv 9 \pmod{55}$.

2. Find the sequence of pseudorandom numbers generated by the linear congruential method with modulus $m = 9$, multiplier $a = 2$, increment $c = 4$, and seed $x_0 = 3$.

3. State whether each of the following sets is finite, countably infinite, or uncountable. If countably infinite, give a one-to-one correspondence between the set of positive integers and the given set.

   (a) $\{x \in \mathbb{Z} : x < 20\}$.
   (b) $\{x \in \mathbb{Z} : |x| < 20\}$.
   (c) The odd negative integers.
   (d) $\{x \in \mathbb{R} : 0 < x < 2\}$
   (e) $S = A \times \mathbb{Z}^+$, where $A = \{1, 2\}$.
   (f) The set of complex numbers $\mathbb{C}$.

4. Let $d \in \mathbb{Z}^+$. The ordered d-tuple $(n_1, n_2, \ldots, n_d)$ is called a *lattice point* when $n_1, n_2, \ldots, n_d$ are all integers ($\in \mathbb{Z}$). Show that the set of all lattice points (for the given d) is countable.

5. Prove or disprove

   (a) The set of all finite subsets of $\mathbb{Z}^+$ is countable.
   (b) The set of all finite subsets of $\mathbb{R}^+$ is countable.
   (c) The set of all infinite subsets of $\mathbb{Z}^+$ is countable.
   (d) The set of all functions $f : \mathbb{Z}^+ \to \{0, 1\}$ is uncountable.

6. Show a construction of:

   (a) Three infinite pair-wise disjoint subsets of $\mathbb{Z}^+$.
   (b) An infinite partition of $\mathbb{Z}^+$ consisting of infinite sets.
   (c) A uncountably infinite partition of $\mathbb{R}$ consisting of countably infinite sets. [Hint: Consider $\mathbb{Z}$...]

7. Does there exist a partition of $\mathbb{R}$ that is an uncountably infinite partition that consists of uncountably infinite sets? If so, clearly explain how to construct such a partition, otherwise prove that such a partition can not exist.

8. In this question you will prove an equivalent (and far cleaner!) definition of infinity:
   A set C is infinite iff there exists a bijection from C onto a proper subset of itself, $B \subset C$.

   (a) Prove one direction: If there exists a bijection from C onto $B \subset C$, then C is infinite.

(b) Prove the converse: If C is infinite, then there exists a bijection from C onto a proper subset of itself.

[Hint: Show that there exists a set $T \subset C$ such that T is countable. Define a bijection $f : \mathbb{N} \to T$ and use f to build a bijection from C to $C \setminus \{a\}$ where $a \in C$.]