CS40 Winter 2021 Homework #4

January 26, 2021

Notes

- You may work with a partner in order to understand the problems and discuss how to approach them. If you do so, write clearly on your assignment the name of the student you collaborated with.
- Justify your answers!
- Please re-read the "Conduct" section in the class syllabus.
- No late submissions! Turn-in what you have by the deadline.

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1. What is the power set of $S = \{a, b, \{c, d\}\}$? (List all elements.)

2. Prove or give a counterexample:
   If x is a real number, then $\lceil x \rceil - \lfloor x \rfloor = 1$ iff x is not an integer.

3. Let f be the function from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = \lfloor x + 1 \rfloor$. Find

   (a) $f^{-1}(\{1\})$,

   (b) $f^{-1}(\{x \mid 0 < x < 1\})$,

   (c) $f^{-1}(\{x \mid x > 4\})$.

4. Determine whether or not each of these functions from $\mathbb{Z}$ to $\mathbb{Z}$ is injective, surjective, bijective, or neither (explain your answers):

   (a) $f(n) = n$

   (b) $f(n) = n^2$

   (c) $f(n) = |n|$

   (d) $f(n) = -n$

   (e) $f(n) = \lfloor \frac{n}{2} \rfloor$

5. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be real-valued functions.

   (a) Given $f(x) = x + 1$ and $g(x) = 2x + 1$. Is $f \circ g$ injective, surjective, bijective, or neither? Is it invertible? If so, find the inverse.

   (b) Find a pair of functions f, g (from $\mathbb{R}$ to $\mathbb{R}$) such that $f \circ g$ is invertible, but $g \circ f$ is not.

6. Let $S \subseteq A$, $T \subseteq A$, and $f : A \to B$.

   (a) Prove $f(S \cup T) = f(S) \cup f(T)$.

   (b) Prove $f(S \cap T) \subseteq f(S) \cap f(T)$.

   (c) Does $f(S \cap T) = f(S) \cap f(T)$? Prove or provide a counterexample.

7. Sequences and summations.

   (a) Compute $\sum_{n=10}^{100} 3$.

   (b) Compute $\sum_{n=0}^{8} (2^{n+1} - 2^n)$.

   (c) Let $S = \{-1, 3, 4, 9\}$. Compute

   $$\sum_{s \in S \cap \mathbb{Z}^+} s \qquad \text{and} \qquad \prod_{s \in S} s$$

   (d) Find a closed-form expression for

   $$\sum_{m=1}^{n} \frac{1}{m(m+1)}$$

   [Hint: Consider rewriting the sum in the form of $\sum_{m=1}^{n} (a_m - a_{m-1})$, which we studied in class.]

   (e) Let a_n be a sequence as follows:
   $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, \ldots$
   That is, we start with $a_1 = 1$ and then each integer m is repeated m times exactly. Compute

   $$\sum_{k=1}^{\infty} \frac{1}{a_k 2^{a_k}}$$

   [Hint: Rewrite the sum by considering how may times each a_k appears.]

8. Suppose a, b, c, d are four real numbers with $a < b$ and $c < d$. Construct an explicit bijection between the set of real numbers in the interval $a < x < b$, and the interval $c < x < d$.

9. Convert

   (a) $(70007)_8$ to its binary expansion

   (b) $(111110001)_2$ to its hexadecimal expansion

   (c) $(1301)_{10}$ to its binary expansion

   (d) $(1010111101)_2$ to its octal expansion

   (e) Suppose using base b, we have $(d04)_b = (22)_b + (32)_b$, where d is some digit. Find b and d. Show your work.

10. Suppose a and b are two numbers with $ab = p^9 q^9 r^9$ and $\text{lcm}(a, b) = p^5 q^5 r^6$ where p, q, r are distinct primes. Find $\gcd(a, b)$.

11. If $a = 2^3 5^4 7^2$ and $b = 2^2 3^3 5^2 7^3 19$, compute the following quantities. You can leave your answers in product form.

    (a) $\gcd(a, b)$

    (b) $\operatorname{lcm}(a, b)$

    (c) the smallest $x \in \mathbb{Z}^+$ such that $ax \equiv 0 \pmod{b}$

12. Martin correctly multiplied two octal numbers, but forgot to write down the most significant digit d of the product. Can you help him?

$$(12345)_8 \times (54321)_8 = (d\,17743365)_8$$

    Show your work.

13. Prove that n is odd if and only if 1 appears an odd number of times in the base 3 expansion of n.