CS40 Winter 2021 Homework #2

January 16, 2021

Notes

- You may work with a partner in order to understand the problems and discuss how to approach them. If you do so, write clearly on your assignment the name of the student you collaborated with.
- Please re-read the "Conduct" section in the class syllabus.
- No late submissions! Turn-in what you have by the deadline.

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1. Prove the following equivalences by using a series of logical equivalences. Justify each step by stating which rule is applied.
   (a) $\neg[r \vee (q \wedge (\neg r \rightarrow \neg p))] \equiv \neg r \wedge (p \vee \neg q)$
   (b) $[q \wedge (p \rightarrow \neg q)] \rightarrow \neg p \equiv \mathbf{T}$
2. Express the following propositions in Conjunctive Normal Form:
   (a) $p \vee q$
   (b) $p \rightarrow (q \wedge r)$
   (c) $\neg(\neg p \vee q) \vee (r \rightarrow \neg s)$
3. Express the negation of the following sentences in English, as simply as you can (using quantifiers, and do not start with "It is not the case that ...")
   (a) Every student in the class likes movies.
   (b) Some student has been to every state except Alaska.
   (c) No student has ever been to a castle in Germany.
   (d) There is a student in the class who has been in every classroom on this campus.
   (e) Some students like movies and some students like operas.
4. Negate "Some integer is negative and all integers are positive." Write your answer in English, try to simplify.
5. Let $P(x, y)$ be the statement "Student x has taken class y," where the domain for x consists of all students in your class and for y consists of all computer science courses at your school. Express each of these quantifications in English:

(a) $\exists x \exists y P(x, y)$
(b) $\exists x \forall y P(x, y)$
(c) $\forall x \exists y P(x, y)$
(d) $\exists y \forall x P(x, y)$
(e) $\forall y \exists x P(x, y)$
(f) $\forall x \forall y P(x, y)$

6. Write the following facts about real numbers in predicate logic (do not use the uniqueness quantifier, $\exists!$):

   (a) "Given a number, there is a number smaller than it."
   (b) "Given a number, there is a unique number which is greater by one than it."
   (c) "Given a non-zero number, its square is positive."
   (d) "The product of two positive numbers is positive."

7. Let
   $C(x)$: "x is a Computer Science major",
   $M(y)$: "y is a math course", and
   $T(x, y)$: "x is taking y",
   where x represents students and y represents courses. Write the following statement in English, using the predicates:

$$\forall x \; \exists y \; [C(x) \rightarrow M(y) \land T(x, y)]$$

8. Rewrite each of the following statements so that the negations appear only within predicates (i.e., no negation is outside a quantifier or an expression involving logical connectives):

   (a) $\neg \forall x \forall y [P(x, y) \rightarrow Q(x, y)]$
   (b) $\neg \forall x [\exists y \forall z P(x, y, z) \lor \exists z \forall y \neg Q(x, y, z)]$

9. Find a common domain for the variables x, y and z for which the statement $\forall x \forall y [(x \neq y) \rightarrow \forall z ((z = x) \lor (z = y))]$ is true and another domain for which it is false.

10. Prove the following propositions.

    (a) If x and y are integers and xy is even, then x is even or y is even.
    (b) If x is a positive integer, then x is even if and only if $3x + 2$ is even.

11. Use proof by contradiction to prove the following:

    (a) In a room of 13 people, 2 or more people have their birthdays in the same month.
    (b) If r is a rational number, then $r + \sqrt{2}$ is irrational.
    (c) An even perfect square cannot be of the form $4k + 1$.

12. Prove that if n is any integer not divisible by 5, then its square is of the form $5k + 1$ or $5k + 4$ (for example $13^2 = 5 \cdot 33 + 4$ and $9^2 = 5 \cdot 16 + 1$).