Combinatorics

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1 Preliminaries

We start first with *counting*. Counting is simply the study of the size of sets with elements sharing a common property, for example, the count of possible outcomes to some experiments.

The most basic properties of counting are the *addition principle*: Given $A_1, A_2, \ldots, A_n$ pairwise-disjoint sets, the number of elements in the union is

$$|A_1 \cup A_2 \cup \ldots A_n| = |A_1| + |A_2| + \ldots + |A_n| = \sum_{j=1}^{n} |A_j|$$

And similarly, the *multiplication principle*: If we do n experiments, where an experiment j has m_j possible outcomes, and all the experiments are unrelated and independent (the outcome of one experiment does not affect the outcome of another), then the total number of all possible outcomes is

$$m_1 \cdot m_2 \cdot \ldots \cdot m_n = \prod_{j=1}^{n} m_j$$

We already did some trivial counting, most notably, we have concluded that an injection from a set with n elements into a set with $m < n$ elements is not possible. Likewise, it is not possible to surject a set with m elements onto a set with $n > m$ elements. And, as an immediate conclusion, a bijection can only exist between (finite) sets with the same count of elements. Thus, as we know very well by now, a finite set with n elements is bijective with the index set $\{1, 2, \ldots, n\}$, and in addition we can also embed the well order "$\leq$" from the positive integers into our finite set. We then concentrate our discussion on finite index sets almost exclusively, however, due to the above, everything is immediately applicable to any finite set.

We start with notation:

Definition 1.1. $[n] = \{1, 2, \ldots, n\}$ *denotes the index set with n elements.*

(we previously used the notation $\mathbf{I}_n$, however the above is a more common notation and is simpler to write). Do not confuse $[n]$ with the notation for the equivalence class! What we mean should be clear from context.

Definition 1.2. *As before, $2^{[n]} = \mathcal{P}([n])$ is the powerset of the index set, and denote $\begin{bmatrix} n \\ k \end{bmatrix} = \{A \in 2^{[n]} : |A| = k\}$, i.e. the set of all subsets of size k.*

Definition 1.3. $[n]^k = \underbrace{[n] \times [n] \times \ldots \times [n]}_{k\text{-times}}$ *is the cartesian product over the index set.*

2 Permutations and Combinations

Permutations A permutation of some objects is a particular linear ordering of the objects. For example: Suppose there are 5 fish in a barrel, we pull 3 out and place them, in order, in a line. How many possible outcomes are there to this experiment? This is a very general sort of problem.

Definition 2.1. *A* k-permutation *of* $[n]$ *is an injection* $\phi : [k] \to [n]$*, and a* permutation *of* $[n]$ *is a bijection of* $[n]$*.*

The number of such permutations is counted trivially using elementary counting arguments:

Definition 2.2. *The number of* k*-permutations over* $[n]$ *is*

$$P(n,k) = n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

Remember that by definition $0! = 1! = 1$. We can deduce that the count of permutations over $[n]$ is $n!$, and the count of 1-permutations over $[n]$, that is the count of ways to choose a single object out of a set with n objects, is n.

Example 2.1. *Given three dices coloured red, blue and green. How many possible ways are there to roll the dices, such that none of them give the same number?*

$$P(6,3) = \frac{6!}{3!} = 6 \cdot 5 \cdot 4 = 120$$

Combinations A combination is a selection of objects without order.

Example 2.2. *If the dices in example 2.1 are uncoloured and indistinguishable, then how many ways are there to roll the three dices such that none of dices give the same value?*

Solution. We know there are $P(6,3)$ possible permutations (with order), and given any permutations, there are $P(3,3) = 3!$ ways to permute the dices colours, such that the values stay the same. Therefore, there are $\frac{P(6,3)}{3!}$ ways to perform the experiment, without order, as described. ∎

A combination is then a selection of objects out of a set, without repetition and irregardless of order:

Definition 2.3. *A* k-combination *from* $[n]$ *is an element in* $\begin{bmatrix} n \\ k \end{bmatrix}$.

Following the same reasoning as in example 2.2, we can count the total possible number of combinations over a set:

Definition 2.4. *The number of* k*-combinations over* $[n]$ *is* $(0 \leq k \leq n)$

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$$

That is, $\left|\begin{bmatrix} n \\ k \end{bmatrix}\right| = C(n,k)$.

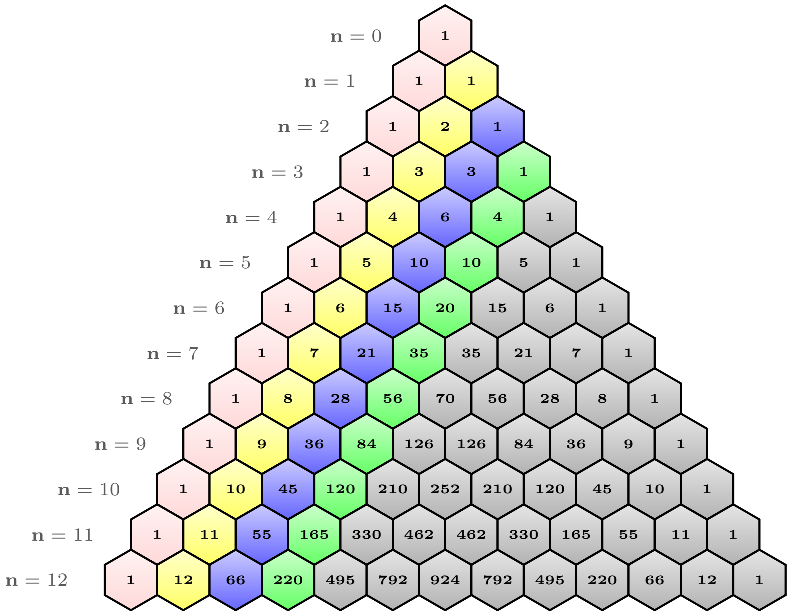


Figure 1: Pascal's Triangle. The numbers inside the hexagons are $C(n, k)$ with n being the row index (listed) and k being the column's index, starting with $k = 0$ and till $k = n$.

2.1 Binomial Coefficient

Listing all the possible k-combinations overs $[n]$ for the first few k, n we get a familiar structure (figure 2.1), known as Pascal's Triangle. The entries in Pascal's triangle are the count of k-combinations over $[n]$, i.e. $C(n, k)$. Notice the symmetry at each row around $k = \frac{n}{2}$, and also note that each entry in Pascal's triangle is generated by adding the two entries directly above it. That symmetry is obvious from the analytic expression for $C(n, k)$, and we will prove the second conjecture:

Lemma 2.1. *$C(n, k) = C(n-1, k-1) + C(n-1, k)$.*

Proof. Consider $[n]$, without loss of generality we consider two types of k-combinations over $[n]$: Those that contain n and those that do not (i.e. are k-combinations over $[n-1]$).

There are $C(n-1, k)$ subsets of size k of $[n]$ that do not contain n. A subset of size k that does contain n can be considered as a an arbitrary subset of size $k-1$ over $[n-1]$, unified with $\{n\}$, and there are $C(n-1, k-1)$ such subsets.

Notice that the sets of subsets that contain n is disjoint form the set of subsets that does not contain n. Therefore, there are $C(n-1, k)+C(n-1, k-1)$ to choose a k-combination from $[n]$. □

The values in Pascal's triangle are known as *binomial coefficients*, they arise when we construct a polynomial that is a sum of two terms (i.e. a binomial) to an integer power.

Definition 2.5. *The* binomial coefficients*, denoted as* $\binom{n}{k}$ *and read as "n choose k", are the coefficients of the binomial expansion:*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Theorem 2.1 (Binomial Theorem)**.** *For every non-negative integers* n, k*,* $\binom{n}{k} = C(n,k)$*. That is, the binomial coefficients are the count of ways to choose a* k*-combination from* $[n]$*.*

Proof. By induction on n. Easy to check for $n = 0, 1, 2$ (do so!).

Assume for $n-1$, i.e. we assume the following holds

$$(x+y)^{n-1} = \sum_{k=0}^{n-1} C(n-1,k) x^{n-k-1} y^k$$

and we would like to prove for n. Then,

$$(x+y)^n = (x+y)(x+y)^{n-1} = (x+y) \sum_{k=0}^{n-1} C(n-1,k) x^{n-k-1} y^k$$

(complete the proof!) □

And we conclude that $\left| \begin{bmatrix} n \\ k \end{bmatrix} \right| = \binom{n}{k}$, which serves to explain the notation.

Examples

Example 2.3. *How many different six-digit numbers can be constructed from the digits* $1, 2, 3$*?*

Solution. 3^6 ■

Example 2.4. *Suppose* n *balls, each of different colour, are in a barrel; we pull out* k *of them, placing them in a line as we do so. How many outcomes are possible? What if we place them in a bag?*

Solution. With order (in a line): $P(n,k) = \frac{n!}{(n-k)!}$;
without order (in a bag): $C(n,k) = \binom{n}{k}$. ■

Example 2.5. *How many ways are there to line up six people so that a particular pair of people are not adjacent?*

Solution. There are $6!$ permutations on six people, however that counts the (illegal) permutations where the particular pair are adjacent. Let the people in that pair be a and b. How many permutations have a and b adjacent? There are 2 options, we place a immediately after b, or we place b immediately after a. Either way, we consider the pair ab or ba as a single object, and permute it together with the other 4 people, there $5!$ permutations, thus $2 \cdot 5!$ ways to arrange the six people such that the pair is adjacent. Therefore, final answer is $6! - 2 \cdot 5!$. ■

Example 2.6. *The Department of Computer Science has nineteen faculty members, of whom ten are women.*

- *How many committees of five can be formed?*
- *How many committees of five can be formed if at least two members must be women and professors Smith and Klaus (both man) refuse to serve together?*
- *How many committees of five (total) can be formed if the committee includes a president and vice-president?*
- *How many committees of five (total), with a president and vice-president, can be formed, if the president must be a woman?*

Solution.

- $\binom{19}{5}$.
- By cases:
  1. Exactly 5 women: $\binom{10}{5}$.
  2. Exactly 4 women: $\binom{10}{4} \times 9$.
  3. Exactly 3 women: $\binom{10}{3} \times \left(\binom{9}{2} - 1\right)$.
  4. Exactly 2 women: $\binom{10}{2}$ for the women. Need to choose 3 men. There are $\binom{7}{2}$ committees with Prof. Smith but not Klaus, likewise $\binom{7}{2}$ with Klaus but not Smith, $\binom{7}{3}$ without both. Total: $\binom{10}{2}\left(2\binom{7}{2} + \binom{7}{3}\right)$.

  Final answer is the sum of all possiblities: $\binom{10}{5} + 9\binom{10}{4} + \binom{10}{3} \times \left(\binom{9}{2} - 1\right) + \binom{10}{2}\left(2\binom{7}{2} + \binom{7}{3}\right)$.
- 19 ways to choose a president, 18 ways to choose a vice-president, and $\binom{17}{3}$ to choose the rest. $19 \times 18 \times \binom{17}{3}$ total.
- 10 ways to choose a president, 18 ways to choose a vice-president, and $\binom{17}{3}$ to choose the rest. $10 \times 18 \times \binom{17}{3}$ total.

■

Example 2.7. *How many positive factors does* $2 \cdot 3^7 \cdot 5^2 \cdot 23$ *has?*

Solution. This is equivalent to choosing the exponents for each prime in the factorization. There are two ways to choose the exponent for 2: 2^0 or 2^1. Similarly, there are eight ways to choose the exponent for 3, etc. Therefore, there are $2 \cdot 8 \cdot 3 \cdot 2 = 96$ ways to choose the exponents, each giving a distinct positive factor. ■

Example 2.8. *Derive the identities*

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \qquad \text{and} \qquad 2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Solution. The first follows from definition 2.5 by setting $y = 1$, and the second follows similarly by setting $x = y = 1$. ■

Example 2.9. *Six people are to sit around a round table with six seats. How many possible sitting arrangements are there when:*

1. *The seats are numbered (and therefore distinguishable).*
2. *The seats are unnumbered and indistinguishable.*
3. *Two sitting arrangements are considered equivalent, if every person has the same neighbours.*

Solution.

1. This is choosing a permutation (with order), i.e. $6!$.
2. The seats are indistinguishable, therefore given a permutation of a sitting arrangements we can rotate the people around the table and the arrangement remains the same. There are 6 ways to do this "rotation" as there are 6 chairs. Therefore $\frac{6!}{6} = 5!$.
3. If we only care about neighbours, then our previous answer counts the equivalence classes twice: clockwise and counter-clockwise. Therefore there are $\frac{5!}{2}$ equivalence classes.

■

Example 2.10. *In how many ways can six hardcover and four paperback books (all books are different) be arranged on a shelf? In how many ways can they be arranged if no two paperbacks can be adjacent?*

Solution. First, we simply choose a permutation on all books, i.e. $10!$.

For the second part, we can permute the hardcover books, and there are $6!$ ways to do so. Now, we need to select spaces for the paperback books, we can place at most one before all the hardcover books, at most one after, and at most one between every adjacent pair of hardcover books. Those are 7 potential spaces, and we need to select 4 from them, that is we need to choose a 4-permutation out of the set of 7 spaces. There are $P(7, 4) = \frac{7!}{3!} = 7 \cdot 6 \cdot 5 \cdot 4$ ways to do so.
That is, there are $\frac{6!7!}{3!}$ ways to arrange the books as required. ■

3 Selection with Repetition

So far we have been mostly counting choices made without repetition (with the exception of multiplication principle). Permutations and combinations are typically described as experiments where we draw a ball from a box, and the implied rule is that we do not place it back. If the ball were to be returned back to the box, or if there were multiple identical balls in the box, then a different combinatorial problem would ensue.

For example, if there are 4 balls of different colours in a box, and we pull 3 balls, without repetition (without returning the balls back), then there are $4 \cdot 3 \cdot 2$ ways to pull a 3-permutation of those balls. However, if a box contains one red ball and one green ball and no other balls and we pull 3 balls with repetition (putting the ball back after each pull), then we can pull 2^3 possible permutations. As another alternative, consider the case where we have 2 blue and 2 black balls in a box and we pull out 3 balls without returning the balls to the box. This is still a "with repetition" because there are indistinguishable elements in the box. A "with repetition" problem can always be formulated as a "without repetition" problem, where instead we have a finite or unlimited count of the same object.

Thus, our focus in this section is on problems where multiple indistinguishable objects are put in a box. This can be formalised mathematically using a multiset, a set where repetitions are allowed:

Definition 3.1. *A* multiset *is the ordered pair* (A, m_A) *where* A *is a set known as the* underlying set, *and* $m_A : A \to \mathbb{Z}^+ \cup \{\infty\}$ *is the* multiplicity *function, i.e. the function that counts the number of appearance of each element of* A.

When $m_A(a) = \infty$*, then we say that the element* $a \in A$ *appears an infinite amount of times.*

An element in a multiset, then, can appear a positive integer count of times or infinitely many times. Usually, problems where elements appear infinitely many times are easier, because they simply reduce to the multiplication principle:

Example 3.1. *How many* k*-permutations can be arranged over the multiset* $\{a_1, a_2, \ldots, a_n\}$ *with* $m(a_j) = \infty$*, i.e. all elements* a_j *in the multiset appear infinitely many times?*

Solution. This is the same question as how many string of length k can be formed over the alphabet $\Sigma = \{a_1, a_2, \ldots, a_n\}$, and, by the multiplication principle, the answer is $\left|\Sigma^k\right| = n^k$ (note that this is simply $\left|[n]^k\right|$). ∎

We are also interested in counting combinations of a multiset, that is selection without order. Such a combination is then a subset of the multiset, and we define such a subset formally:

Definition 3.2. *A subset of a multiset* (A, m_A) *is the multiset* (B, m_B)*, where* $B \subseteq A$ *and* $\forall b \in B(m_B(b) \leq m_A(b))$*. That is, a subset of a multiset is a multiset that does not contain an element more times than the original multiset.*

Example 3.2. *Given a multiset with* $A = \{a_1, a_2, \ldots, a_n\}$ *and* $m_A(A) = \{\infty\}$*, how many* k*-combinations over that multiset can be selected?*

Solution. This is equivalent to asking how many submultisets, with a total number of elements k, exist. We can reformulate the problem in more familiar terms: Say we have $n-1$ indistinguishable hardcover books, n types of distinguishable papercover books (but infinitely many of each) and we have $k+n-1$ bookshelf spaces. We are going to arrange the hardcover books in arbitrary order, and fill the empty spaces with the papercover books as follows, all the spaces up to the first hardcover book are filled with the first type of papercover book (i.e. a_1), the spaces between the first and second hardcover books are filled with the second type of papercover book (a_2), etc. This uniquely determines a submultiset with k elements. There are $\binom{k+n-1}{n-1} = \binom{k+n-1}{k}$ (equality by symmetry) ways to do so. ∎

So far, we can summarise: The number of ways to choose k out of n objects...

- with order and without repetition, is $P(n, k)$;

- with order and with repetition, is n^k;
- without order and without repetition, is $\binom{n}{k}$;
- without order and with repetition, is $\binom{k+n-1}{k}$.

Let's consider now experiments where the number of repetitions of the elements is not infinite. To simplify the notation, we will also write a multiset as $\{a \times \infty, b \times 3, c \times 2\}$ to indicate a multiset with element a appearing infinitely many times, b appearing thrice and c appearing twice.

Example 3.3. *How many permutations of the multiset* $\{a_1 \times m_1, a_2 \times m_2, \ldots, a_n \times m_n\}$ *are there?*

Solution. Let $M = \sum_{j=1}^{n} m_j$. We are asked how many distinct M-permutations can be selected. There are $M!$ ways to arrange all M elements, however that over-counts as not all elements are distinguishable. Such a permutation is equivalent to all permutations where we only permute the objects a_1, and there are $m_1!$ ways to do so. Likewise, permuting the a_2-s does not change the permutation, and there are $m_2!$ ways to do so, etc. We can conclude, that there are $\frac{M!}{m_1! m_2! \ldots m_n!}$ distinct permutations over the multiset. ■

Definition 3.3. $\binom{n}{k_1\ k_2\ \ldots\ k_m} = \frac{n!}{\prod_{j=1}^{m} k_j!}$ *is the* multinomial coefficient*.*

Notice, that $\binom{n}{k} = \binom{n}{n\ (n-k)}$, that is the binomial coefficient is a special case of the multinomial coefficient.

Distributing objects It is also common to describe combinatorial problems as distributing objects into boxes:

Example 3.4. *In how many ways can we distribute* $[n]$ *into the boxes numbered* $[k]$ *(distinguishable objects into distinguishable boxes), such that box* j *has exactly* m_j *elements?*

Solution. Let $M = \sum m_j$. Because M might be less than n, we set up an additional box $k+1$ for discarded elements, i.e. into which we put $m_{k+1} = n-M$ elements. Then, this is the same problem as choosing a permutation of the multiset $\{a_1 \times m_1, a_2 \times m_2, \ldots, a_k \times m_k, a_{k+1} \times m_{k+1}\}$. There are, by design, a total of n elements in that multiset, therefore the element at index j in a permutation simply describes into which box element $j \in [n]$ goes.
There are $\binom{n}{m_1\ m_2\ \ldots\ m_k\ (n-M)}$ ways to permute the multiset. ■

Example 3.5. *In how many ways can we distribute* $\{a \times n\}$ *into boxes numbered* $[k]$ *(indistinguishable objects into distinguishable boxes)?*

Solution. This is the same as asking for the non-negative integer solutions to the equation $x_1 + x_2 + \ldots + x_k = n$, i.e. distributing n 1-s into k (distinguishable) boxes. Or equivalently, define the multiset $\{1_{x_1} \times \infty, 1_{x_2} \times \infty, \ldots, 1_{x_k} \times \infty\}$ (i.e. the elements $1_{x_1}, 1_{x_2}, \ldots, 1_{x_k}$ that appear infinitely many times) and choose a n-combination. There are $\binom{n+k-1}{n}$ ways to do so. ■

Distributing objects into indistinguishable boxes are harder problems.

Examples

Example 3.6. *Find the number of integer solutions to* $x_1+x_2+x_3 = 10$*, when* $x_1, x_2 \geq 0$ *and* $x_3 \geq 1$*.*

Solution. This is the same as looking for the number of non-negative integer solutions to $x_1+x_2+(x_3-1) = 9$. And, we know that there are $\binom{9+3-1}{9} = \frac{11\cdot 10}{2}$ ways to do so. ■

Example 3.7. *How many unique permutations of the letters in the word "kinnikinnick" are there?*

Solution. There are 3 k-s, 1 c, 4 i-s and 4 n-s. Therefore, $\binom{12}{1\ 3\ 4\ 4} = \frac{12!}{3!4!4!}$ permutations. ■

Example 3.8. *A box contains 18 balls, numbered 1-6, 3 identical balls with each number. We draw 4 balls, how many possible outcomes are there to this experiment, with order and without order?*

Solution. With order: select a 4-permutation from $\{a_1 \times \infty, a_2 \times \infty, \ldots, a_6 \times \infty\}$, i.e. 6^4. We need to subtract illegal permutations: that is, permutations with 4 identical objects. There 6 such permutations, there $6(6^3 - 1)$.

Without order: same idea, just with 4-combinations, then $\binom{4+6-1}{4} - 6$. ■

4 The Pigeonhole Principle

This rather simple but useful principle is oft involved in showing that two possible different values are in fact the same.

Theorem 4.1. *Suppose that* m *objects are distributed amongst* n *boxes, such that* $m > n$*. Then, at least one box will have more than one object.*

Proof. By contradiction: If the premise holds but each box has at most one object, then the total count of objects in the boxes is less or equal to n, but $n < m$, contradiction. □

This can be used in many ways.

Example 4.1. *Let* E *be a set consisting of* 11 *double-digit positive integers. Then* E *must have a pair of elements whose (positive) digit difference is the same.*

Example 4.2. *If there four pairs of socks in a drawer, picking five socks ensures that at least one pair is chosen.*

Example 4.3. *Let* $a_1, a_2, \ldots, a_n$ *be positive integers. Show that there exists integers* k, m*, such that the partial sum* $a_k + a_{k+1} + \ldots + a_{k+m}$ *is divisible by* n *(*$k, k+m \leq n$*).*

Solution. Denote the partial sums:

$$
\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
\ldots \\
s_n &= a_1 + a_2 + \ldots + a_n
\end{aligned}
$$

i.e. $s_k = \sum_{j=1}^{k} a_j$. Assume none of those sums is divisible by n (otherwise, we are done). Consider the remainders $r_k = s_k \bmod n$. By assumption, $1 \leq r_k < n$, however there are n sums, therefore, by the pigeonhole principle, there exists $i < j$ such that $s_i \equiv s_j \pmod{n}$. Then, $s_j - s_i \equiv 0 \pmod{n}$, and $s_j - s_i = a_{i+1} + a_{i+2} + \ldots + a_j$, as required. ■

An immediate generalization is the follows:

Corollary 4.1. *Given n boxes and $m \geq n(r-1)+1$ objects, for some integer $r > 0$. Then, if we distribute the objects amongst the boxes, one box will have at least r objects.*

We leave the proof for the reader.

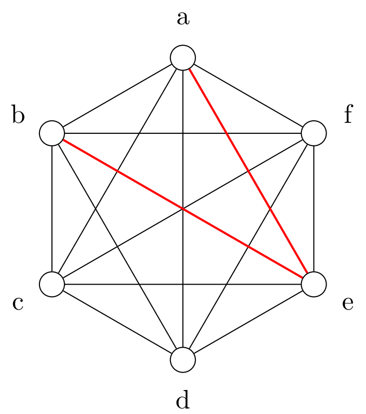
Example 4.4. *What is the minimum required number of students in a CS40 class to ensure that at least five students get the same grade (from the possible grades A,B,C,D,E,F)?*

Solution. There are six grade, i.e. $n = 6$, and we want to ensure that $r = 5$ students get the same grade. Then, $m = 6 \cdot 4 + 1 = 25$ students would be enough to ensure that at least 5 students indeed must get the same grade.

To show that it is the minimum we consider 24 students, however we can partition those students into 4 students get an A, 4 students get a B, etc. With no five students having the same grade. ■

Example 4.5. *Suppose 6 people are gathered together; then either 3 of them are mutually acquainted, or 3 of them are mutually unacquainted.*

Solution. Denote the 6 people $\{a, b, c, d, e, f\}$, and consider the following graph. We colour the edge between a pair of people black if the pair is unacquainted, otherwise red.



In the sample graph above, e is acquainted with a and b, and everyone else is pair-wise unacquainted. We need to show that such a graph will, in general, always give rise to a black or red triangle.

Consider (without loss of generality) person a. a will have 5 edges adjacent to it, and there are 2 colours. Then, if we were to distribute those edges into coloured boxes, by the generalized pigeonhole principle (Corollary 4.1), with $n = 2$, $r = 3$ and $m = 2(3-1)+1 = 5$, at least three of those edges will have colour C (red or black), and let the people connected by those edges be x, y, z. If any pair amongst x, y, z shares an edge coloured C, then we have formed a triangle with colour C. Otherwise, x, y, z form a triangle with the other colour. ■

5 Inclusion-Exclusion Principle

Let's review a familiar problem: How many non-negative integer solutions are there to the equation $x_1 + x_2 + x_3 = 8$, where we also want to ensure that $x_1 < 3$, $x_2 < 4$ and $0 < x_3 < 5$. We have solved a similar problem in example 3.6, however now our multiset has elements which appear a limited count of times. First, we remove the lower-bound, as before, by considering the equation $y_1 + y_2 + y_3 = 7$, with non-negative integers $y_1 < 3, y_2 < 4, y_5 < 4$, and note that given a solution to this equation $x_1 = y_1, x_2 = y_2, x_3 = y_3 + 1$ is a solution to the former equation.

We know that the number of non-negative solutions to the equation $y_1 + y_2 + y_3 = 7$, ignoring upper-bounds, is a selection without order and with repetition, i.e. $\binom{7+3-1}{7} = 36$. However, remembering the bounds $y_1 < 3, y_2 < 4, y_3 < 4$, the above over-counts solutions that should be disqualified, like $y_1 = 7, y_2 = y_3 = 0$. How many solutions have $y_1 \geq 3, y_2 \geq 0, y_3 \geq 0$? This is a problem we know how to solve: $y_1 + y_2 + y_3 = 7$ with $y_1 \geq 3$ is equivalent to $(y_1 - 3) + y_2 + y_3 = 4$ and there are $\binom{4+3-1}{4} = 15$ non-negative integer solutions. Similarly, there are $\binom{3+3-1}{3} = 10$ non-negative solutions when we limit $y_1 \geq 0, y_2 \geq 4, y_3 \geq 0$, and a further $\binom{3+3-1}{3} = 10$ non-negative solutions when $y_1 \geq 0, y_2 \geq 0, y_3 \geq 4$.

Those solutions should be subtracted from the total solutions we found and we get $36 - 15 - 10 - 10 = 1$, but would that be it? No, because now we are under-counting! We have subtracted the solution $y_1 = 3, y_2 = 4, y_3 = 0$ twice. So, we now need to add back solutions where $y_1 \geq 3, y_2 \geq 4$, and this is again a problem that we know how to solve: there is exactly one non-negative integer solution to $(y_1 - 3) + (y_2 - 4) + y_3 = 0$. Similarly, there a single solution when $y_1 \geq 3, y_3 \geq 4$, and no solutions when $y_2 \geq 4, y_3 \geq 4$. All those solutions have been subtracted twice, and we add them back to our total. We are still not done, because now we are over-counting solutions with $y_1 \geq 3, y_2 \geq 4, y_3 \geq 4$, and that is because we added those solutions once at first, removed them thrice at the second step, and then added them three times again at the third step. But, there are 0 such solutions. Finally, we get that the number of possible solutions is $36 - 15 - 10 - 10 + 1 + 1 = 3$.

This was rather troublesome and we had a simple equation with a few variables a small numbers. Thankfully, this process can be generalized and we will now formalise it: Let U be the universe that consists of all non-negative integer solutions to $y_1 + y_2 + y_3 = 7$. Let S_1, S_2, S_3 be the sets of solutions with $y_1 \geq 3$,$y_2 \geq 4$,$y_3 \geq 4$, respectively. We would like to find

$$S = \left|S_1^C \cap S_2^C \cap S_3^C\right|$$

and we concluded that

$$S = |U| - |S_1| - |S_2| - |S_3| + |S_1 \cap S_2| + |S_1 \cap S_3| + |S_2 \cap S_3| - |S_1 \cap S_2 \cap S_3|$$

Generalized to any finite intersection of complements, we get the inclusion-exclusion principle:

Theorem 5.1 (The Inclusion-Exclusion Principle). *Let $S_1, S_2, \dots, S_n \subseteq U$ be sets with a universe U. Then,*

$$\left|\bigcap_{j=1}^{n} S_j^C\right| = |U| + \sum_{k=1}^{n} (-1)^k \sum_{I \in \begin{bmatrix} n \\ k \end{bmatrix}} \left|\bigcap_{m \in I} S_m\right|$$

Proof. First, we will show that every $a \in \bigcap_{j=1}^{n} S_j^C$ is counted once by the right-hand-side. Clearly, $a \notin S_m$ for every $m \in [n]$, therefore it is counted by, and only by, $|U|$.

The second part is harder, we would like to show that every $a \notin \bigcap_{j=1}^{n} S_j^C$ is not counted by the right-hand-side. a must be in some S_m-s, choose $M \in \begin{bmatrix} n \\ l \end{bmatrix}$, such that $m \in M$ iff $a \in S_m$ and $l > 0$ by assumption. Consider the intersection $\bigcap_{m \in I} S_m$ in the formula above. a is an element in that intersection iff $I \subseteq M$.

By the above, we only need to consider the subsets $I \subseteq M$, with $|I| = k$, and note that $1 \leq k \leq l = |M|$. How many subsets of M of size k are there? I.e. how many k-combinations are there over $[l]$? $\binom{l}{k}$. Thus, for a given k, the sum

$$\sum_{I \in \begin{bmatrix} n \\ k \end{bmatrix}} \left|\bigcap_{m \in I} S_m\right|$$

counts the element a exactly $\binom{l}{k}$ times. a is also counted once by $|U|$, therefore, a is counted $1 + \sum_{k=1}^{l} (-1)^k \binom{l}{k}$ times in total. This can be rewritten as

$$1 + \sum_{k=1}^{l} (-1)^k \binom{l}{k} = \sum_{k=0}^{l} (-1)^k \binom{l}{k} = 0$$

The last equality follows directly from the definition of the binomial coefficient (definition 2.5) by setting $x = -1, y = 1$, thus completing the proof. □

Using the above we get the following:

Corollary 5.1.

$$\left|\bigcup_{j=1}^{n} S_j\right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{I \in \begin{bmatrix} n \\ k \end{bmatrix}} \left|\bigcap_{m \in I} S_m\right|$$

which you will also prove on your homework.

Examples

Example 5.1. *Find the number of positive integers not exceeding 100 that are not divisible by 5 or 7.*

Solution. Let $U = [100]$ be the universe. Denote by $A_5 = 5\mathbb{Z} \cap U$ as the set of integers divisible by 5, and $A_7 = 7\mathbb{Z} \cap U$ as the set of integers divisible by 7. Then, noting that $A_5 \cap A_7 = 35\mathbb{Z} \cap U$,

$$\left|A_5^C \cap A_7^C\right| = |U| - |A_5| - |A_7| + |A_5 \cap A_7| = 100 - 20 - 14 + 2 = 68$$

■

Example 5.2. *Among 18 students in a room, 7 study mathematics, 10 study physics, and 10 study computer science. Also, 3 study computer science and physics, 4 study computer science and mathematics, and 5 study physics and mathematics. We know that a single student studies all three subjects. How many of these students study none of those three subjects?*

Solution. Let S_m, S_p, S_{cs} be the set of students that study mathematics, physics and computer science, respectively. Then,

$$\begin{aligned}\left|S_m^C \cap S_p^C \cap S_{cs}^C\right| =&|U| - |S_m| - |S_p| - |S_{cs}| \\ &+ |S_m \cap S_p| + |S_m \cap S_{cs}| + |S_p \cap S_{cs}| \\ &- |S_m \cap S_p \cap S_{cs}| \\ =&18 - 7 - 10 - 10 + 5 + 4 + 3 - 1 = 2\end{aligned}$$

■

6 Exercises

Exercise 6.1. *In how many ways can five men and five women be arranged in a line for a photograph, such that that men and women alternate?*

Solution. There are 5! ways to permute the men, and similarly 5! ways to permute the women. There are two ways to interlace the two permutations, *mwmwmwmwmw* or *wmwmwmwmwm*, giving a final answer of $2 \cdot (5!)^2$. ■

Exercise 6.2. *Combinatorially explain the identity* $\binom{3n}{3} = n^3 + 6n\binom{n}{2} + 3\binom{n}{3}$.

Solution. Assume we have a $3n$-element set A. $\binom{3n}{3}$ is the number of ways to choose a 3-combination out of A. The right-hand-side contains expressions that are related to choosing combinations from an n-element set. Let's force a partition into n-element sets on A: Colour all elements in A with three colours, red, green and blue, and number the elements of the same colour $[n]$. There are $\binom{n}{3}$ ways to choose 3 elements of a particular colour, therefore there are $3\binom{n}{3}$ ways to choose 3 elements of the same colour. There are $3\binom{n}{2} \cdot 2\binom{n}{1} = 6n\binom{n}{2}$ ways to choose 2 elements of the same colour, as well as an additional element of another colour. Finally, there are $\binom{n}{1}^3 = n^3$ ways to choose 3 elements, one of each colour. ■

Exercise 6.3. *Joe's Pizzeria offers two styles of crust and the following optional toppings: extra cheese, pepperoni, sausage, mushrooms, green peppers, artichokes, onions, and anchovies. Joe claims that he offers over* 500 *different pizzas. Is this true?*

Solution. Yes. 2^8 choices to toppings, and 2 choices of crust, i.e. $2^9 = 512$ pizza choices. ■

Exercise 6.4. *Each of the nine unit squares of a* 3×3*-square is coloured randomly red or blue, each with probability* $\frac{1}{2}$*. Determine the probability that none of the four* 2×2*-squares is completely red.*

Solution. This is asking, what is the ratio of the number of legal colourings of the 3×3-square to the number of all possible colourings, and there are $|U| = 2^9 = 512$ possible colourings.

Let A_1 denote the colourings where the first 2×2 square is completely red. There are $|A_1| = 2^5$ such colourings. And we similarly define A_2, A_3, A_4. We proceed by considering the colouring where adjacent squares are entirely red, i.e. $|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_4| = |A_3 \cap A_4| = 2^3$. Likewise, colourings where diagonal squares are entirely red, $|A_1 \cap A_4| = |A_2 \cap A_3| = 2^2$.

When 3 squares are entirely red, there are always 2 colourings for each combination of 3 squares, and there is only a single colouring where all squares are red.

Therefore $\left|A_1^C \cap A_2^C \cap A_3^C \cap A_4^C\right| = 512 - 4 \cdot 2^5 + 4 \cdot 2^3 + 2 \cdot 2^2 - \binom{4}{3} \cdot 2 + 1.$ ■

Exercise 6.5. *Prove that* $(a+b)^3 \equiv a^3 + b^3 \pmod 3$ *where* $a, b \in \mathbb{Z}$.
Is it true that $(a+b)^{10} \equiv a^{10} + b^{10} \pmod{10}$*?*

Solution. By the binomial theorem,

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \equiv a^3 + b^3 \pmod 3$$

However,

$$0 = (-1+1)^10 \not\equiv (-1)^{10} + 1^{10} \equiv 2 \pmod{10}$$

■

Exercise 6.6. *Prove the identity*

$$\binom{n+2}{k+1} = \binom{n}{k+1} + 2\binom{n}{k} + \binom{n}{k-1}$$

Solution. For an arbitrary $x \in \mathbb{R}$, consider

$$\begin{aligned}(1+x)^{n+2} = (1+x)^2(1+x)^n &= (1+x)^2 \sum_{j=0}^{n} \binom{n}{j} x^j \\ &= (1+2x+x^2) \sum_{j=0}^{n} \binom{n}{j} x^j \\ &= \sum_{j=0}^{n} \binom{n}{j} \left(x^j + 2x^{j+1} + x^{j+2}\right)\end{aligned}$$

where we used the Binomial theorem. Using the Binomial theorem again:

$$(1+x)^{n+2} = \sum_{j=0}^{n+2} \binom{n+2}{j} x^j$$

therefore

$$\sum_{j=0}^{n+2} \binom{n+2}{j} x^j = \sum_{j=0}^{n} \binom{n}{j} \left(x^j + 2x^{j+1} + x^{j+2}\right)$$

because x is arbitrary, the equality implies that the coefficients are equal. On the left-hand-side $\binom{n+2}{k+1}$ is the coefficient of the term x^{k+1}. On the right-hand-side, the coefficients of x^{k+1} are $\binom{n}{k+1} + 2\binom{n}{k} + \binom{n}{k-1}$, proving the identity. ■